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ON 2-SWELLING TOPOLOGICAL GROUPS

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Dedicated to the 60th birthday of Igor Guran

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A topological group G is called *2-swelling* if for any compact subsets $A, B \subset G$ and elements $a, b, c \in G$ the inclusions $aA \cup bB \subset A \cup B$ and $aA \cap bB \subset c(A \cap B)$ are equivalent to the equalities $aA \cup bB = A \cup B$ and $aA \cap bB = c(A \cap B)$. We prove that an (abelian) topological group G is 2-swelling if each 3-generated (resp. 2-generated) subgroup of G is discrete. This implies that the additive group $\mathbb{Q}$ of rationals is 2-swelling and each locally finite topological group is 2-swelling.

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Топологічна група G називаються *2-набряклою*, якщо для довільних компактних підмножин $A, B \subset G$ і елементів $a, b, c \in G$ включення $aA \cup bB \subset A \cup B$ та $aA \cap bB \subset c(A \cap B)$ еквівалентні рівностям $aA \cup bB = A \cup B$ та $aA \cap bB = c(A \cap B)$. Доведено, що (абелева) топологічна група G є 2-набряклою, якщо кожна 3-породжена (відп. 2-породжена) підгрупа групи G дискретна. Звідси випливає, що адитивна група $\mathbb{Q}$ раціональних чисел є 2-набряклою і кожна локально скінченна топологічна група 2-набряклою.

In this paper we give a partial solution the following question [2] of Alexey Muranov posted at MathOverFlow on July 24, 2013.

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Problem 1 (Muranov). *Let A, B be compact subsets of a Hausdorff topological group G such that $aA \cup bB \subset A \cup B$ and $aA \cap bB \subset c(A \cap B)$ for some elements $a, b, c \in G$. Is $aA \cup bB = A \cup B$ and $aA \cap bB = c(A \cap B)$?*

For example, for any real numbers $u < v < w < t$ the closed intervals $A = [u, w]$ and $B = [v, t]$ satisfy the equalities $(a + A) \cup (b + B) = A \cup B$ and $(a + A) \cap (b + B) = c + (A \cap B)$ for $a = t - w$, $b = u - v$ and $c = v + w - u - t$.

First we observe that Muranov's Problem 1 has an affirmative answer for finite sets $A, B \subset G$.

Proposition 2. *Let A, B be finite subsets of a group G and $aA \cup bB \subset A \cup B$ and $aA \cap bB \subset c(A \cap B)$ for some elements $a, b, c \in G$. Then $aA \cup bB = A \cup B$ and $aA \cap bB = c(A \cap B)$.*

Proof. Evaluating the cardinality of the corresponding sets, we conclude that

$$|A \cup B| \geq |aA \cup bB| = |aA| + |bB| - |aA \cap bB| \geq \\ \geq |A| + |B| - |c(A \cap B)| = |A| + |B| - |A \cap B| = |A \cup B|,$$

$|A \cup B| = |aA \cup bB|$ and $|aA \cap bB| = |c(A \cap B)|$, which imply the desired equalities $aA \cup bB = A \cup B$ and $aA \cap bB = c(A \cap B)$. □

The Muranov's problem can be reformulated as a problem of 2-parametric generalization of the classical Swelling Lemma, see [1, 1.9].

Theorem 3 (Swelling Lemma). *For any compact subset A of a Hausdorff topological semigroup S and any element $a \in S$ the inclusion $aA \supset A$ is equivalent to the equality $aA = A$.*

In this paper we shall give some partial affirmative answers to Muranov's Problem 1. First, we introduce appropriate definitions.

Definition 4. A topological group G is called *2-swelling* if for any compact subsets $A, B \subset G$ and elements $a, b, c \in G$ the inclusions $aA \cup bB \subset A \cup B$ and $aA \cap bB \subset c(A \cap B)$ are equivalent to the equalities $aA \cup bB = A \cup B$ and $aA \cap bB = c(A \cap B)$.

Definition 5. A topological group G is called *weakly 2-swelling* if for any compact subsets $A, B \subset G$ and elements $a, b \in G$ with $aA \cap bB = \varnothing$ the inclusion $aA \cup bB \subset A \cup B$ implies $aA \cup bB = A \cup B$ and $A \cap B = \varnothing$.

The following trivial proposition implies that the class of (weakly) 2-swelling topological groups is closed under taking subgroups.

Proposition 6. *Let $h : G \to H$ be a continuous injective homomorphism of topological groups. If the topological group H is (weakly) 2-swelling, then so is the group G.*

A subgroup H of a group G is called *n-generated* for some $n \in \mathbb{N}$ if H is generated by set of cardinality $\le n$. It can be shown that a topological group G is Hausdorff if each 1-generated subgroup of G is discrete. The main result of this paper is the following theorem.

Theorem 7. *An (abelian) topological group G is*

- *2-swelling if each 3-generated (resp. 2-generated) subgroup of G is discrete;*
- *weakly 2-swelling if each 2-generated (resp. 1-generated) subgroup of G is discrete.*

Corollary 8. *For every $n \in \mathbb{N}$ the group $\mathbb{Q}^n$ is 2-swelling and the group $\mathbb{R}^n$ is weakly 2-swelling.*

Corollary 9. *Each locally finite topological group is 2-swelling.*

We recall that a group G is *locally finite* if each finite subset $F \subset G$ generates a finite subgroup. In the proof of Theorem 7 we shall use the following lemma.

Lemma 10. *Let $K \subset G$ be a compact subset of a topological group and H be a discrete subgroup of G. Then the set $K \cap H$ is finite and $\sup_{x\in G} |K \cap Hx| \le |KK^{-1} \cap H| < \infty$. If the discrete subgroup H is closed in G, then for every $n \in \mathbb{N}$ the set $\{x \in G : |K \cap Hx| \ge n\}$ is closed in G.*

Proof. First we show that the intersection $K \cap H$ is finite. Assuming that $K \cap H$ is infinite, we can choose a sequence $(x_n)_{n\in\omega}$ of pairwise distinct points of $K \cap H$. By the compactness of K the sequence $(x_n)_{n\in\omega}$ has an accumulation point $x_\infty \in K$.

Since the subgroup $H \subset G$ is discrete, the unit 1 of H has a neighborhood $U_1 \subset G$ such that $U_1 \cap H = \{1\}$. Choose a neighborhood $V_1 \subset G$ of 1 such that $V_1V_1^{-1} \subset U_1$. Since x_∞ is an accumulation point of the sequence $(x_n)_{n\in\omega}$, the neighborhood V_1x_∞ of x_∞ contains two distinct points x_n, x_m of the sequence. Then $x_nx_m^{-1} \in V_1V_1^{-1} \cap H \subset U_1 \cap H = \{1\}$ and hence $x_n = x_m$, which contradicts the choice of the sequence (x_k). So, $K \cap H$ is finite. The set KK^{-1} being a continuous image of the compact space $K \times K$ is compact too, which implies that $KK^{-1} \cap H$ is finite.

Next, we prove that $\sup_{x\in G} |K \cap Hx| \le |KK^{-1} \cap H| < \infty$. Given any point $x \in G$ with $K \cap Hx \ne \emptyset$, choose a point $y \in K \cap Hx$ and observe that $|K \cap Hx| =$

$|K \cap Hy| = |Ky^{-1} \cap H| \le |KK^{-1} \cap H|$. Then $\sup_{x\in G} |K \cap Hx| \le |KK^{-1} \cap H| < \infty$.

Now assume that the discrete subgroup H is closed in G. To see that for every $n \in \mathbb{N}$ the set $G_n = \{x \in G : |K \cap Hx| \ge n\}$ is closed in G, choose any element $x \in G$ with $|K \cap Hx| < n$. Then the set $F = Kx^{-1} \cap H$ has cardinality $|F| = |K \cap Hx| < n$. It follows that $H \setminus F$ is a closed subset of G, disjoint with the compact set Kx^{-1}. For every $y \in K$ we can find a neighborhood $O_y \subset G$ of y and a neighborhood $O_{x,y} \subset G$ of x such that $O_y O_{x,y}^{-1} \cap (H \setminus F) = \varnothing$. By the compactness of K, the open cover $\{O_y : y \in K\}$ of K has finite subcover $\{O_y : y \in E\}$ (here $E \subset K$ is a suitable finite subset of K). Then the neighborhood $O_x = \bigcap_{y\in E} O_{x,y}$ of x has the property: $KO_x^{-1} \cap (H \setminus F) = \varnothing$, which equivalent to $KO_x^{-1} \cap H \subset F$ and implies $O_x \cap G_n = \varnothing$, witnessing that the set G_n is closed in G. □

Remark 11. It is well-known that any discrete subgroup H of a Hausdorff topological group G is closed in G. In general case this is not true: for any discrete topological group H and an infinite group G endowed with the anti-discrete topology, the subgroup $H \times \{1_G\}$ of $H \times G$ is discrete but not closed in G.

The following theorem implies Theorem 7, and is the main technical result of the paper.

Theorem 12. *Let $A, B \subset G$ be two compact subsets of a topological group G such that $aA \cup bB \subset A \cup B$ and $aA \cap bB \subset c(A \cap B)$ for some points $a, b, c \in G$. The equalities $aA \cup bB = A \cup B$ and $aA \cap bB = c(A \cap B)$ hold if either the subgroup H_3 generated by the set $\{a, b, c\}$ is discrete or for some subset $T \subset \{a, b, c\}$ with $\{a, b\} \subset T$, $\{a, c\} \subset T$ or $\{b, c\} \subset T$ the subgroup H_2 generated by T is discrete and closed in G, and H_2 is normal in the subgroup H_3.*

Proof. The proof splits into 2 parts.

I. The subgroup H_3 generated by the set $\{a, b, c\}$ is discrete. By the compactness of the sets A, B, aA, bB, for every $x \in G$ the sets $(aA \cup bB) \cap H_3x$ and $(A \cup B) \cap H_3x$ are finite (see Lemma 10), so we can evaluate their cardinality: $|(aA \cup bB) \cap H_3x| \le |(A \cup B) \cap H_3x|$ and $|aA \cap bB \cap H_3x| \le |c(A \cap B) \cap H_3x| = |A \cap B \cap c^{-1}H_3x| = |A \cap B \cap H_3x|$. Next, observe that

$$\begin{aligned}
|(A \cup B) \cap H_3x| &\ge |(aA \cup bB) \cap H_3x| = \\
&= |aA \cap H_3x| + |bB \cap H_3x| - |(aA \cap bB) \cap H_3x| \ge \\
&\ge |A \cap H_3x| + |B \cap H_3x| - |A \cap B \cap H_3x| = |(A \cup B) \cap H_3x|,
\end{aligned}$$

which implies that $|(A \cup B) \cap H_3x| = |(aA \cup bB) \cap H_3x|$ and $|(aA \cap bB) \cap H_3x| = |A \cap B \cap H_3x|$ and hence $(A \cup B) \cap H_3x = (aA \cup bB) \cap H_3x$ and $(aA \cap bB) \cap H_3x = A \cap B \cap H_3x$. Finally,

$$aA \cup bB = \bigcup_{x\in G}(aA \cup bB) \cap H_3x = \bigcup_{x\in G}(A \cup B) \cap H_3x = A \cup B \text{ and}$$

$$aA \cap bB = \bigcup_{x\in G}(aA \cap bB) \cap H_3x = \bigcup_{x\in G}(A \cap B) \cap H_3x = A \cap B.$$

II. The subgroup H_3 is not discrete but for some set $T \subset \{a,b,c\}$ with $\{a,b\} \subset T$, $\{a,c\} \subset T$ or $\{b,c\} \subset T$ the subgroup H_2 generated by T is closed and discrete in G and H_2 is normal in H_3. In this case the quotient group H_3/H_2 is not discrete. Let t be the unique element of the set $\{a,b,c\}\setminus T$. It follows that $H_3 = \bigcup_{n\in\mathbb{Z}} H_2t^n$. The normality H_2 in H_3 implies that $H_2t = tH_2$. Depending on the value of t three cases are possible.

(a) First consider the case of $t = a$ and $\{b,c\} \subset T$. In this case for every $x \in G$ the inclusions $aA \cap bB \cap H_2x \subset c(A \cap B) \cap H_2x$ and $(aA \cup bB) \cap H_2x \subset (A \cup B) \cap H_2x$ imply $|aA \cap bB \cap H_2x| \le |c(A \cap B) \cap H_2x| = |(A \cap B) \cap c^{-1}H_2x| = |(A \cap B) \cap H_2x|$ and

$$\begin{aligned}
&|(A \cup B) \cap H_2x| \ge |(aA \cup bB) \cap H_2x| = \\
&= |aA \cap H_2x| + |bB \cap H_2x| - |(aA \cap bB) \cap H_2x| \ge \\
&\ge |A \cap a^{-1}H_2x| + |B \cap b^{-1}H_2x| - |(A \cap B) \cap H_2x| = \qquad (1)\\
&= |A \cap a^{-1}H_2x| - |A \cap H_2x| + |A \cap H_2x| + |B \cap H_2x| - |(A \cap B) \cap H_2x| = \\
&= |A \cap a^{-1}H_2x| - |A \cap H_2x| + |(A \cup B) \cap H_2x|.
\end{aligned}$$

Consequently,

$$|A \cap H_2a^{-1}x| \le |A \cap H_2x| \text{ and } |A \cap H_2x| \le |A \cap H_2ax| \qquad (2)$$

for every $x \in G$.

By Lemma 10, for every $x \in G$ the number $\alpha_x = \max_{n\in\mathbb{Z}} |A \cap H_2a^n x|$ is finite, so we can find a number $n_x \in \mathbb{Z}$ such that $|A \cap H_2a^{n_x}x| = \alpha_x$. By Lemma 10, the set $G_\alpha = \{g \in G : |A \cap H_2g| \ge \alpha_x\}$ is closed in G.

The inequalities (2) guarantee that $\alpha_x = |A \cap H_2a^{n_x}x| \le |A \cap H_2a^m x| = \alpha_x$ for all $m \ge n_x$ and hence $\bigcup_{m\ge n_x} H_2a^m x \subset G_\alpha$. Taking into account that the quotient group H_3/H_2 is not discrete and is generated by the coset H_2a, we conclude that the set $\bigcup_{m\ge n_x} H_2a^m x$ is dense in H_3, and hence $H_3 \subset G_\alpha$, which means that $|A \cap H_2 a^m x| = \alpha_x$ for all $m \in \mathbb{Z}$ and hence all inequalities in (2) and (1) turn into equalities. In particular, $|(aA \cup bB) \cap H_2 x| = |(A \cup B) \cap H_2 x|$ and $|(aA \cap bB) \cap H_2 x| = |(A \cap B) \cap H_2 x| = |c(A \cap B) \cap H_2 x|$ for all $x \in G$. Combining these equalities with the inclusions $(aA \cup bB) \cap H_2 x \subset (A \cup B) \cap H_2 x$ and $(aA \cap bB) \cap H_2 x \subset c(A \cap B) \cap H_2 x$, we conclude that $(aA \cup bB) \cap H_2 x = (A \cup B) \cap H_2 x$ and $(aA \cap bB) \cap H_2 x = c\,(A \cap B) \cap H_2 x$ for all $x \in G$ and hence

$$aA \cup bB = \bigcup_{x \in G} (aA \cup bB) \cap H_2 x = \bigcup_{x \in G} (A \cup B) \cap H_2 x = A \cup B \text{ and}$$

$$aA \cap bB = \bigcup_{x \in G} (aA \cap bB) \cap H_2 x = \bigcup_{x \in G} c(A \cap B) \cap H_2 x = c(A \cap B).$$

(b) The case of $t = b$ and $\{a, c\} \subset T$ can be considered by analogy with the case (a).

(c) Finally we consider the case of $t = c$ and $\{a, b\} \subset H_2$. Observe that for every $x \in G$ the inclusion $(aA \cap bB) \cap H_2 x \subset c(A \cap B) \cap H_2 x$ implies

$$|aA \cap bB \cap H_2 x| \le |c(A \cap B) \cap H_2 x| = |A \cap B \cap c^{-1} H_2 x| = |A \cap B \cap H_2 c^{-1} x|.$$

On the other hand,

$$\begin{aligned} &|A \cap B \cap H_2 c^{-1} x| \ge |aA \cap bB \cap H_2 x| = \\ &= |aA \cap H_2 x| + |bB \cap H_2 x| - |(aA \cup bB) \cap H_2 x| \ge \\ &\ge |A \cap a^{-1} H_2 x| + |B \cap b^{-1} H_2 x| - |(A \cup B) \cap H_2 x| = \\ &= |A \cap H_2 x| + |B \cap H_2 x| - (|A \cap H_2 x| + |B \cap H_2 x| - |A \cap B \cap H_2 x|) = \\ &= |A \cap B \cap H_2 x|. \end{aligned} \tag{3}$$

Therefore,

$$|A \cap B \cap H_2 x| \le |A \cap B \cap H_2 c^{-1} x|. \tag{4}$$

By Lemma 10, for every $x \in G$ the number $\Delta_x = \max_{n \in \mathbb{Z}} |A \cap B \cap H_2 c^n x|$ is finite, so we can find a number $n_x \in \mathbb{Z}$ such that $|A \cap B \cap H_2 c^{n_x} x| = \Delta_x$. By Lemma 10, the set $G_\Delta = \{g \in G : |A \cap B \cap H_2 g| \ge \Delta_x\}$ is closed in G.

The inequality (4) guarantees that

$$\Delta_x = |A \cap B \cap c^{n_x} x| \le |A \cap B \cap H_2 c^m x| \le \Delta_x$$

for all $m \le n_x$ and hence $\bigcup_{m \le n_x} H_2 c^m x \subset G_\Delta$. Taking into account that the quotient group H_3/H_2 is not discrete and is generated by the coset $H_2 c$, we conclude that the set $\bigcup_{m\le n_x} H_2c^m x$ is dense in H_3, and hence $H_3 \subset G_\Delta$. Then $|A\cap B\cap H_2c^m x| = \Delta_x$ for every $m\in\mathbb{Z}$, which implies that all inequalities in (4) and (3) turn into equalities. In particular, we get the equalities $|A\cap B\cap H_2c^{-1}x| = |aA\cap bB\cap H_2x|$ and $|(aA\cup bB)\cap H_2x| = |(A\cup B)\cap H_2x|$, which imply the equalities $aA\cap bB\cap H_2x = c(A\cap B)\cap H_2x$ and $(aA\cup bB)\cap H_2x = (A\cup B)\cap H_2x$ holding for all $x\in G$. Then

$$aA\cup bB = \bigcup_{x\in G}(aA\cup bB)\cap H_2x = \bigcup_{x\in G}(A\cup B)\cap H_2x = A\cup B \quad\text{and}$$

$$aA\cap bB = \bigcup_{x\in G}(aA\cap bB)\cap H_2x = \bigcup_{x\in G} c(A\cap B)\cap H_2x = c(A\cap B).$$

□

Corollary 13. *Let $A, B\subset G$ be two compact subsets of a topological group G such that $aA\cup bB\subset A\cup B$ and $aA\cap bB=\varnothing$ for some points $a,b\in G$. If the cyclic subgroup H_a generated by a is closed and discrete in G and H_a is normal in the subgroup $H_{a,b}$ generated by the set $\{a,b\}$, then $aA\cup bB = A\cup B$ and $A\cap B=\varnothing$.*

Proof. Apply Theorem 12 with a point $c\in\{a,b,1_G\}$. □

The most intriguing problem about (weakly) 2-swelling groups concerns the real line (and the circle).

Problem 14 (Muranov)**.** *Is the additive group $\mathbb{R}$ of real numbers 2-swelling? Is the group $\mathbb{R}/\mathbb{Z}$ weakly 2-swelling?*

Theorem 12 implies that if $\mathbb{R}$ is not 2-swelling, then it contains two compact sets $A, B\subset\mathbb{R}$ such that $(a+A)\cup(b+B)\subset A\cup B$ and $(a+A)\cap(b+B)\subset c+(A\cap B)$ for some non-zero real numbers a,b,c such that all fractions $\frac{a}{b}$, $\frac{a}{c}$, $\frac{b}{c}$ are irrational. The following proposition shows that such sets A, B necessarily have positive Lebesgue measure.

Proposition 15. *Assume that $A, B\subset G$ are two non-empty compact subsets of $\mathbb{R}$ such that $aA\cup bB\subset A\cup B$ for some non-zero real numbers a,b with irrational fraction $\frac{a}{b}$. Then $A+b\mathbb{Z}=\mathbb{R}=B+a\mathbb{Z}$ and hence the sets A,B have positive Lebesgue measure.*

Proof. Given any point $x_0\in A$, for every $n\in\mathbb{N}$ let

$$x_n = \begin{cases} a+x_{n-1} & \text{if } x_{n-1}\in A,\\ b+x_{n-1} & \text{otherwise,}\end{cases} \quad\text{and}\quad s_n = \begin{cases} a & \text{if } x_{n-1}\in A,\\ b & \text{otherwise.}\end{cases}$$

Using the inclusion $(a + A) \cup (b + B) \subset A \cup B$ we can prove by induction that $\{x_n\} \subset A \cup B$. The compactness of $A \cup B$ implies that the sets $\{n \in \mathbb{N} : s_n = a\}$ and $\{n \in \mathbb{N} : s_n = b\}$ are infinite. Consider the cyclic subgroup $b\mathbb{Z}$ generated by b and let $q_b : \mathbb{R} \to \mathbb{R}/b\mathbb{Z}$ be the quotient homomorphism. Observe that for every $n \in \mathbb{N}$ $q_b(x_n) = q_b(x_n)$ if $s_n = b$ and $q_b(x_n) = q_b(a + x_{n-1}) = q_b(a) + q_b(x_{n-1})$ if $x_n \in A$. Since the set $\{n \in \mathbb{N} : s_n = a\}$ is infinite and the fraction $\frac{a}{b}$ is irrational, the set $q_b(\{x_n\}_{n\in\mathbb{N}})$ is dense in the quotient group $\mathbb{R}/b\mathbb{Z}$. The definition of the numbers s_n guarantees that $q_b(\{x_n\})_{n\in\mathbb{N}}) \subset q_b(A)$. Now the compactness of A guarantees that $q_b(A) = \mathbb{R}/b\mathbb{Z}$ and hence $A + b\mathbb{Z} = \mathbb{R}$.

By analogy, we can prove that $B + a\mathbb{Z} = \mathbb{R}$. □

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