THE ALGEBRAIC STRUCTURE OF A LINEAR FOCKER-PLANCK TYPE KINETIC DYNAMICAL SYSTEM

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The article is devoted to the symmetry analysis of a linear nonuniform Focker-Planck type kinetic dynamical system $u_t = u_{xx} + xu_x + u := K[x; u]$. Making use of the Lie-Backlund symmetry approach new nonuniform and non-autonomous hierarchies of symmetries are constructed.

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1 Introduction

We study the symmetries of the nonuniform Focker-Planck type kinetic dynamical system

$$u_t = u_{xx} + xu_x + u := K[x;u], \tag{1.1}$$

on M, where $M \subset C^\infty(\mathbb{R};\mathbb{R})$ is a Schwartz type functional submanifold, $x \in \mathbb{R}$ is the spatial variable and $t \in \mathbb{R}_+$ is the evolution parameter.

Our analysis is based completely on the Lie-Backlund symmetry approach [1, 2, 4], which makes it possible to construct for equation (1.1) new hierarchies of nonuniform and non-autonomous symmetries.

2 Symmetry and recursion structure analysis

We start from observing that equation (1.1) on the manifold M can be representable in the split form as

$$K[x;u] = \alpha_1[u] + 2\beta_0[x;u], \tag{2.1}$$

where

$$\alpha_1[u] := u_{xx}, \qquad \beta_0[x;u] := \frac{1}{2}(xu_x + u). \tag{2.2}$$

Moreover, as it is well known [1], the expression $\beta_0[x;u] := \frac{1}{2}(xu_x + u)$ is a nonuniform symmetry of the flow

$$u_{t_1} := u_{xx} = \alpha_1[u], \tag{2.3}$$

satisfying the important Lie commutator condition

$$[\beta_0, \alpha_1] = \alpha_1. \tag{2.4}$$

Herewith, we can easily construct the suitable uniform and non-autonomous Lie-Backlund symmetries of flow (2.1) from those of flow (2.3), which we will consider from now as a generating one. Now we will make use of the elementary fact from [1, 4] that all of linear operators $\tilde{\Lambda}_n := \partial^n : T(M) \to T(M)$, $n \in \mathbb{Z}$, acting in the tangent space $T(M)$, are the "*recursion*"operators for the generative flow (2.3), that is the Lie derivatives

$$L_{\alpha_1}\tilde{\Lambda}_n := d\tilde{\Lambda}_n/dt_1 - [\alpha_1^{'}, \tilde{\Lambda}_n] = 0, \tag{2.5}$$

for all $n \in \mathbb{Z}$, where L_X denotes the Lie derivative [2] along the vector field $X : M \to T(M)$ and the sign "$\prime$" denotes the standard Frechet derivative. In particular,

$$\tilde{\Lambda}_1 = \partial, \quad \tilde{\Lambda}_2 = \partial^2, \quad \tilde{\Lambda}_3 = \partial^3 \tag{2.6}$$

and so on. Taking further into account (2.4) one easily obtains that

$$L_K\tilde{\Lambda}_1 = L_{\alpha_1+2\beta_0}\tilde{\Lambda}_1 = L_{\alpha_1}\tilde{\Lambda}_1 + 2L_{\beta_0}\tilde{\Lambda}_1 = 0 + \partial = \tilde{\Lambda}_1. \tag{2.7}$$

Recalling that the Lie derivative L_K is here a derivation in the space *End* $T(M)$, we can define the expression

$$\hat{\Lambda}_{[1]} := e^{-t}\tilde{\Lambda}_1 = e^{-t}\partial, \tag{2.8}$$

satisfying the generalized "*determining*"recursion condition

$$(\partial/\partial t + L_K)\hat{\Lambda}_{[1]} = 0 \tag{2.9}$$

for our dynamical system (1.1). As a corollary, we obtain that all expressions

$$K_j^{(1)} := \hat{\Lambda}_{[1]}^j K[x; u] \tag{2.10}$$

for $j \in \mathbb{Z}$ are also the nonuniform-non-autonomous symmetries of flow (1.1).

The same way we can construct many other recursion operators for (1.1) from (2.7), in particular, expressions

$$\hat{\Lambda}_{[n]} := \hat{\Lambda}_{[1]}^n = e^{-nt}\partial^n \tag{2.11}$$

for all $n \in \mathbb{Z}$ are recursive operators for (1.1) too.

Doing similarly, one can succeed in finding from (2.7) other recursion operators as

$$\begin{aligned} \hat{\Lambda}_{(1)} \; &: = \hat{\Lambda}_{[1]} = e^{-t}\partial, \quad \hat{\Lambda}_{(2)} = \partial^2 + x\partial, \\ \hat{\Lambda}_{(3)} &= e^{t}(\partial + x), \\ \hat{\Lambda}_{(4)} &= e^{2t}(\partial^2 + 2x\partial + x^2 + 1). \end{aligned} \tag{2.12}$$

For instance, it is easy to calculate that

$$\begin{aligned} (\partial/\partial t + L_K)(\tilde{\Lambda}_2 + x\tilde{\Lambda}_1) &= L_K(\tilde{\Lambda}_2 + x\tilde{\Lambda}_1) = L_K\tilde{\Lambda}_2 + \\ +L_K(x\tilde{\Lambda}_1) &= 2\partial^2 - 2\partial^2 = 0, \end{aligned} \tag{2.13}$$

that is the expression

$$\hat{\Lambda}_{(2)} = \tilde{\Lambda}_2 + x\tilde{\Lambda}_1 = \partial^2 + x\partial \tag{2.14}$$

is a new recursion operator for flow (1.1).

Similarly,

$$L_K(\tilde{\Lambda}_2+2x\tilde{\Lambda}_1+f(x;t)I) = -2\tilde{\Lambda}_2-2f_x(x;t)\partial-xf_x(x;t)I-f_{xx}(x;t)I \quad (2.15)$$

since for any scalar multiplication operator $f(x;t)I : T(M) \to T(M)$, $x \in \mathbb{R}$, there holds the general relationship

$$L_K[f(x;t)I] = [f(x;t)I, K'] = 2f_x(x;t)\partial - xf_x(x;t)I - f_{xx}(x;t)I. \quad (2.16)$$

From (2.15) at $f(x;t) = x^2$ one obtains easily that

$$L_K(\tilde{\Lambda}_2 + 2x\tilde{\Lambda}_1 + x^2 + 1) = -2(\tilde{\Lambda}_2 + 2x\tilde{\Lambda}_1 + x^2 + 1), \quad (2.17)$$

which can be equivalently rewritten as

$$(\partial/\partial t + L_K)[e^{2t}(\tilde{\Lambda}_2 + 2x\tilde{\Lambda}_1 + x^2 + 1)] = 0, \quad (2.18)$$

meaning, evidently, that the operator

$$\hat{\Lambda}_{(4)} := e^{2t}(\tilde{\Lambda}_2 + 2x\tilde{\Lambda}_1 + x^2 + 1) = e^{2t}(\partial^2 + 2x\partial + x^2 + 1)$$

is recursive for (1.1). Concerning the operator $\hat{\Lambda}_{(3)} = e^t(\partial + x)$ one finds easily, as above, that

$$L_K\tilde{\Lambda}_1 = -\partial - x = -(\tilde{\Lambda}_1 + x). \quad (2.19)$$

Making use of (2.16) at $f(x;t) = x$, expression (2.19) one rewrites equivalently as

$$(\partial/\partial t + L_K)[e^t(\tilde{\Lambda}_1 + x)] = 0. \quad (2.20)$$

Thereby, owing to (2.20) the operator

$$\hat{\Lambda}_{(3)} := e^t(\tilde{\Lambda}_1 + x) = e^t(\partial + x) \quad (2.21)$$

is also recursive for flow (1.1). The algorithm demonstrated above can be, evidently, continued further for any $n \in \mathbb{Z}$.

3 The symmetry generation

Now we are in a position to construct regularly the sets of nonuniform and non-autonomous symmetries

$$K_j^{(r)} := \hat{\Lambda}_{(r)}^j K, \quad (3.1)$$

where $j \in \mathbb{Z}$, $r = \overline{1,4}$. (We are not writing down them in explicit form leaving these calculations for the interested Reader.)

It is the place here to stress once more that all of the results presented above are obtained regularly making use of the Lie-Bäcklund symmetries and related recursion operators properties in a proper way, as it was just demonstrated.

In the work we have constructed the nonuniform and non-autonomous symmetries of the Focker-Planck type dynamical system (1.1). The method used is based on the splitting trick of the right hand of (1.1) into two parts, satisfying the standard Lie subalgebra condition. The related recursion operators, which can be constructed regularly for any $n \in \mathbb{Z}$, follow easily from the fact that the first split part, mentioned above, as a basic dynamical system, possesses an infinite hierarchy of elementary recursion operators.

4 Conclusion

The nonuniform Focker-Planck type equation (1.1) can be considered as a model for studying much more interesting and complicated nonlinear dynamical systems such as nonuniform Burgers, Korteweg-de Vries and Schrodinger type systems. It is easy to observe that their full analytical treatment fits very deeply to the parametric gradient-holonomic scheme, devised in [4]. Moreover, the complete picture of such dynamical systems can be extracted from the fundamental criterion of so-called Lax type iso-spectrally integrable nonlinear dynamical systems, which is formulated [3, 4] as follows.

Criterion: *Any Lax type iso-spectrally integrable dynamical system* $K : M \to T(M)$ *possesses at least one special infinite-dimensional Lie subalgebra* $\mathcal{G} := \{\alpha_j, \beta_k : j, k \in \mathbb{Z}\}$ *of uniform and non-uniform symmetries isomorphic to the current Lie algebra of the Banach group* $\mathcal{G} = Diff(\mathbb{S}^1) \ltimes D(\mathbb{S}^1)$*, the semi-direct product of the diffeomorphism group* $Diff(\mathbb{S}^1)$ *of the circle* $\mathbb{S}^1$ *and the abelian group* $D(\mathbb{S}^1)$ *of smooth* 2π*-periodic functions. The corresponding recursion operator* $\tilde{\Lambda} : T(M) \to T(M)$ *satisfies the set of determining equations*

$$L_{\alpha_j}\tilde{\Lambda} = 0, \quad L_{\beta_j}\tilde{\Lambda} = \tilde{\Lambda}^{j+1}, \tag{4.1}$$

where vector fields $\alpha_j, \beta_k : M \to T(M)$*,* $j, k \in \mathbb{Z}$*, satisfy the Lie algebra relationships*

$$[K, \alpha_j] = 0, \quad [\alpha_j, \alpha_j] = 0, \quad [\beta_k, \alpha_j] = (j + \varepsilon)\alpha_{j+k} \tag{4.2}$$

for some parameter $\varepsilon \in \mathbb{R}$, *depending on the dynamical system* $K : M \rightarrow T(M)$.

As a corollary, one can construct many infinite hierarchies of non-uniform nonlinear dynamical systems

$$K_j = \alpha_j + \sum_{k \in \mathcal{A}_j} c_{jk} \beta_k \tag{4.3}$$

for any sets of indices $\mathcal{A}_j$, $j \in \mathbb{Z}$, and values $c_{jk} \in \mathbb{R}$, $k \in \mathcal{A}_{.j}$, whose suitable recursion operators $\tilde{\Lambda}_{(j)} : T(M) \rightarrow T(M)$, $j \in \mathbb{Z}$, follow the same scheme, as used above, from the determining equations

$$L_{K_j} \tilde{\Lambda} = \sum_{k \in \mathcal{A}_j} c_{jk} \tilde{\Lambda}^{k+1}. \tag{4.4}$$

Based on equations (4.4) one can obtain the corresponding recursion operators for dynamical systems (4.3) on which we do not stop here in detail. This and related problems will be analyzed elsewhere.

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АЛГЕБРАЇЧНА СТРУКТУРА СИМЕТРІЙ ЛІНІЙНОЇ КІНЕТИЧНОЇ ДИНАМІЧНОЇ СИСТЕМИ ТИПУ ФОККЕРА-ПЛАНКА

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Стаття присвячена симетрійному аналізу лінійної неоднорідної кінетичної динамічної системи типу Фоккера-Планка $u_t = u_{xx} + xu_x + u := K[x;u]$. Грунтуючись на симетрійному підході Лі-Беклунда побудовані нові ієрархії неоднорідних та неавтономних симетрій.