ASYMPTOTIC DIMENSION OF SYMMETRIC POWERS

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The main result of this note is the following one: $\operatorname{asdim} SP^n_G X \leq n \operatorname{asdim} X$, where X is a proper metric space and asdim stands for the asymptotic dimension in the sense of Gromov. The corresponding result is also valid for a generalization of the asymptotic dimension called the Assouad-Nagata asymptotic dimension. By SP^n_G the G-symmetric power functor is denoted.

1 Introduction

The notion of dimension, being one of the most fundamental mathematical notions, has its counterparts in different areas of mathematics, in particular, in the asymptotic topology (see, e.g. [3]). The latter deals with the large scale properties of metric spaces and, more generally, of the so called coarse spaces.

The asymptotic dimension of metric spaces is introduced by M. Gromov [2]. This dimension is a quasi-isometry invariant and therefore can be defined for finitely generated groups. It turned out that the asymptotic dimension plays an important role in the geometric group theory. Therefore, it is of interest to find counterparts of results in the classical (covering) dimension theory for the asymptotic dimension. The aim of this note is to find an estimation of the asymptotic dimension of the G-symmetric powers. The corresponding result for the covering dimension is proved by Basmanov [4].

Let us start with the necessary definitions. Recall that a family $\mathcal{U}$ of subsets in a metric space X (a generic metric is defined by d) is called uniformly bounded if $\operatorname{mesh} \mathcal{U} = \sup\{\operatorname{diam} U \mid U \in \mathcal{U}\} < \infty$. Given $D > 0$,

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we say that a family $\mathcal{U}$ of subsets of X is D-disjoint if, for any $U, V \in \mathcal{U}$, $U \neq V$,

$$d(U, V) = \inf\{d(x, y)|\ x \in U, y \in V\} > D.$$

Definition 1.1. We say that a metric space X is of asymptotic dimension $\leq n$ (denoted $\operatorname{asdim} X \leq n$), if, for every $D > 0$, there exists a uniformly bounded cover $\mathcal{U}$ of X such that $\mathcal{U} = \mathcal{U}^0 \cup \mathcal{U}^1 \cup \cdots \cup \mathcal{U}^n$, where every family $\mathcal{U}^i, i = 0, 1, \ldots, n$, is D-disjoint.

The notion of asymptotic dimension turned out to be of great importance in geometric group theory, analysis, metric geometry and other fields of mathematics.

If is well-known that the asymptotic dimension satisfies the logarithmic law

$$\operatorname{asdim}(X \times Y) \leq \operatorname{asdim} X + \operatorname{asdim} Y$$

(the metric on the product $X \times Y$ is defined by

$$d((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), d(y_1, y_2)\}).$$

For any metric space $X, n \in \mathbf{N}$, we therefore have

$$\operatorname{asdim} X^n \leq n \operatorname{asdim} X. \tag{$*$}$$

In this paper, we prove a generalization of inequality $(*)$, namely, we prove the inequality

$$\operatorname{asdim} SP^n_G X \leq n \operatorname{asdim} X, \tag{$**$}$$

where SP^n_G stands for the G-symmetric power functor.

Recall that, for any subgroup G of the symmetric group S_n, the G-symmetric power $SP^n_G X$ is defined as follows. Denote by $\sim$ the following equivalence relation on $X^n : (x_1, \ldots, x_n) \sim (y_1, \ldots, y_n)$ if there exists $\sigma \in G$ such that $y_i = x_{\sigma(i)}$, for every $i = 1, \ldots, n$. Denote by $[x_1, \ldots, x_n]$ the equivalence class containing $(x_1, \ldots, x_n)$. Then $SP^n_G X = \{[x_1, \ldots, x_n] | (x_1, \ldots, x_n) \in X^n\} = X^n/G$.

The metric $\widehat{d}$ on $SP^n_G X$ is defined as follows:

$$\widehat{d}([x_1, \ldots, x_n], [y_1, \ldots, y_n]) = \min_{\sigma \in G} \max_{1 \leq i \leq n} d(x_i, y_{\sigma(i)}).$$

Note that, in the case of trivial group G, one obtains the l_∞-metric on the space $SP^n_G X = X^n$.

The main result of the paper is a counterpart of Basmanov's theorem on dimension of the symmetric powers as well as another spaces of the form $F(X)$, where F is a functor in the category of compact Hausdorff spaces [5]. Note that, for the case of asymptotic dimension zero, the result is proved in [6]. The following definition corresponds to an equivalent definition of the asymptotic dimension (see [10]).

The notion of the asymptotic dimension was recently modified. Recall that $L(\mathcal{U})$ is a Lebesgue number of a cover $\mathcal{U}$ of a metric space; that $L(\mathcal{U}) > d$ means that every ball of radius d is contained in an element of the family $\mathcal{U}$.

Definition 1.2. The asymptotic Assouad-Nagata dimension of a metric space X does not exceed n, $AN\text{-asdim}\, X \le n$, if there is a $c > 0$ and an $r_0 > 0$ such that for every $r \ge r_0$, there is a cover $\mathcal{U}$ of X such that $\text{mesh}\,\mathcal{U} \le cr$, $L(\mathcal{U}) > r$, and $\mathcal{U}$ has multiplicity $\le n+1$ (the latter means that every point of X belongs to at most $n+1$ elements of $\mathcal{U}$).

2 Preliminaries

Recall that a uniform polyhedron is a polyhedron whose vertices are unit vectors of the Hilbert space l_2. Such a polyhedron is considered as a metric space with the induced metric. Given a uniform polyhedron, K, we say that a map $f\colon X \to |K|$ is uniformly cobounded if, there exists $M > 0$ such that, for every vertex x of K, $\text{diam}\, f^{-1}(St(x)) < M$. (Here, by $St(x)$ we denote the open star of a vertex x).

Recall that a map $f\colon (X,d) \to (Y,\rho)$ is C-Lipschitz (where $C > 0$ is a constant) if

$$\rho(f(x), f(y)) \le C d(x,y),$$

for every $x, y \in X$.

We say that a map f is *Lipschitz* if f is C-Lipschitz, for some $C > 0$. Also, a map is *bi-Lipschitz* if both f and f^{-1} are Lipschitz.

A metric space X is an *absolute neighborhood Lipschitz retract* if for any metric space $Y \supset X$, there exists a Lipschitz retraction $r\colon U \to X$, for some neighborhood U of X in Y.

Let (X,d) be a path connected metric space. The *length-metric* on X is defined as follows. Given $x, y \in X$, and a path $\gamma\colon [a,b] \to X$ connecting x and y, we let

$$L(\gamma) = \sup_{\sigma} \sum_{i=1}^{n} d(\gamma(t_i), \gamma(t_{i-1})),$$

where the supremum is taken over all the subdivisions $\sigma = (t_i)_{i=0}^n$ of the segment $[a,b]$. A path γ is called *rectifiable* if $L(\gamma) < \infty$. The lenght-metric ϱ is defined by the formula

$$\varrho(x,y) = \inf\{L(\gamma) \mid \gamma \text{ is a rectifiable path connecting } x \text{ and } y\}$$

(we tacitly assumed that every two points in X are connected with rectifiable paths).

We will need the following characterization theorem for asymptotic dimension ([2]; see [10] for the detailed proof).

Theorem 2.1. *A metric space X is of asymptotic dimension $\le n$ if and only if, for every $\varepsilon > 0$, there exists a uniform polyhedron K of dimension $\le n$ and an ε-Lipschitz uniformly cobounded map $f\colon X \to |K|$.*

Let us describe a triangulation of $SP^n_G X$, for a polyhedron $X = |K|$ (see [8]). We denote by V the set of vertices of K and by S the set of simplices of K. Let $\le$ be a partial order on V such that $\le$ restricted on any subset $A \in S$ of V is a linear order in A.

We first define a triangulation K^n of X^n (see [8]) as follows. Let V^n be the set of vertices of K^n. Define a partial order $\le$ on V^n by $(v_1,\dots,v_n) \le (w_1,\dots,w_n)$ whenever $v_i \le w_i$ for every $i = 1,\dots,n$.

A subset $T = \{(w_1(t),\dots,w_n(t)) | \ t \in A\}$ is a simplex in K^n if, for any $i \in \{1,\dots,n\}$, the set $\{w_i(t) | \ t \in A\}$ are the vertices (not necessarily distinct) of a simplex in K.

Then the projection $\pi_i\colon X^n \to X$ on the i-th factor is a simplicial map with respect to this triangulation.

By $Sd(K^n)$ we denote, as usual, the barycentric subdivision of the triangulation K^n. Recall that the set B of vertices of $Sd(K^n)$ consists of all the barycenters of the simplices in K^n.

By the definition, $K(n,G)$, a triangulation of $SP^n_G X$ is defined as follows. The set A of vertices of $K(n,G)$ is B/G (the group G acts on B by the permutation of the coordinates). The m-simplices of $K(n,G)$ are in one-to-one correspondence wish the equivalence classes of m-simplices in $Sd(K^n)$. Note that $SP^n_G X$ is an nm-dimensional polyhedron if X is an n-dimensional polyhedron.

In what follows we assume that $m \ge 1$; the case $m = 0$ is considered in [6].

We will need the following statement.

Proposition 2.2. *The identity map* $\mathrm{id}\colon |K| \to |Sd(K)|$ *is bi-Lipschitz.*

Proof. Without loss of generality, one may assume that $|K|$ is connected and bounded with respect to the length-metric (otherwise one may connect every vertex of $|K|$ with a point outside $|K|$ by a one-dimensional segment. Then, as it is well-known, since $|K|$ is a Lipschitz neighborhood retract, the length-metric on $|K|$ is bi-Lipschitz equivalent to the metric induced from l_2.

Therefore, we will consider the length-metrics on $|K|$ and $|Sd(K)|$. Note that, for every simplex σ of K, the map $id|_\sigma$ is Lipschitz and Lipschitz constant C can be chosen to that is does not depend on σ. We conclude that the map id is Lipschitz.

Applying similar arguments to the map $\mathrm{id}^{-1} = \mathrm{id}\colon |Sd(K)| \to |K|$ one concludes that this map is Lipschitz as well. □

Proposition 2.3. *Let $f\colon X \to Y$ be a C-Lipschitz map, where (X, d) and (Y, ϱ) are proper metric spaces. Then the map $SP^n_G f\colon SP^n_G X \to SP^N_G Y$ is C-Lipschitz as well.*

Proof. Let $[x_1, \ldots, x_n], [y_1, \ldots, y_n] \in SP^n_G X$, then there exists $\sigma \in S_n$ such that

$$\hat{\varrho}(SP^n_G f([x_1, \ldots, x_n]), SP^n_G f([y_1, \ldots, y_n])) = \max_i \varrho(f(x_i), f(y_{\sigma(i)})),$$

therefore

$$\begin{aligned}\hat{\varrho}(SP^n_G([x_1, \ldots, x_n]), SP^n_G([y_1, \ldots, y_n])) \le& \max_i Cd(x_i, y_{\sigma(i)}) \\ \le& C\hat{f}([x_1, \ldots, x_n], [y_1, \ldots, y_n])\end{aligned}$$

and we are done. □

3 Main result

The main result of this note is the following

Theorem 3.1. *For any proper metric space X, we have*

$$\operatorname{asdim} SP^n_G X \le n \operatorname{asdim} X.$$

Proof. Let $\operatorname{asdim} X = m$. Let $f\colon X \to |K|$ be a C-Lipschitz map uniformly cobounded map, where K is an m-dimensional uniform polyhedron. The composition map

$$SP^n_G X \xrightarrow{SP^n_G f} SP^n_G |K| \xrightarrow{SP^n_G \mathrm{id}} SP^n_G |Sd(K)|$$

is also C-Lipschitz. We are going to show that this map is uniformly cobounded. Since any open star with respect to the triangulation $Sd(K)$ is a subset of a triangulation K, one have only to demonstrate that the family

$$\{(SP^n_G f)^{-1}((St(x_1)\times\cdots\times St(x_n))/\sim)\mid x_i \text{ is a vertex of } K,\ i=1,\dots,n\}$$

is uniformly bounded. Since the map

$$(x_1,\dots,x_n)\mapsto[x_1,\dots,x_n]\colon X^n\to SP^n_G X$$

is known to be nonexpanding, to this end is sufficient to show that the family

$$\{f^{-1}(St(x_1))\times\cdots\times f^{-1}(St(x_n))\mid x_i \text{ is a vertex of } K,\ i=1,\dots,n\}$$

is uniformly bounded. In its turn, this follows from the fact that

$$\operatorname{diam} f^{-1}(St(x_1))\times\cdots\times f^{-1}(St(x_n))=\max\{\operatorname{diam} f^{-1}(St(x_1))\mid i=1,\dots,n\}$$

and the uniform coboundedness of the map f. By Proposition 2.3, the map f is C-Lipschitz.

It follows from Theorem 2.1 that $\operatorname{asdim} SP^n_G X\le nm$. □

4 Asymptotic Assouad-Nagata dimension

It is known (see [10, Proposition 1.7]) that, for any metric space (X,d), the following are equivalent:

(1) AN-$\operatorname{asdim} X\le n$;

(2) there is a $C>0$ and an $\varepsilon_0>0$ such that for all $\varepsilon<\varepsilon_0$ ($\varepsilon>0$), there is an ε-Lipschitz, C/ε-cobounded map $p\colon X\to P$ to an n-dimensional simplicial complex P.

(Here, a map f into a simplicial complex is called C-*cobounded* if

$$\operatorname{diam} f^{-1}(St(x))\le C,$$

for any vertex x).

Theorem 4.1. *For any proper metric space X, we have*

$$AN\text{-}\operatorname{asdim} SP^n_G X\le n\, AN\text{-}\operatorname{asdim} X.$$

Actually, the proof of of Theorem 3.1 works also in the case of the Assouad-Nagata dimension. We have only to use the characterization of the asymptotic Assouad-Nagata dimension in terms of the maps into polyhedra which is placed above.

5 Remarks and open questions

It is an open question whether the counterpart of this result holds for another functors acting in the asymptotic categories. The metric on the spaces of the form $F(X)$, where F is a normal functor of finite degree (see [1] for the notion of normal functor in the category of compact Hausdorff spaces which served as a model for the corresponding notion in the asymptotic category) is defined in [6]. A possible approach to attacking this problem can be based on results of [9] on triangulation of the spaces of the form $F(X)$, where X is a polyhedron and F a covariant functor of finite degree acting in some topological categories. A particular case of the hypersymmetric power functor is briefly discussed in [7].

Another possible generalization concerns the so called coarse structures. The asymptotic dimension of coarse spaces is considered in [11].

Recently, a hyperbolic dimension hypdim of metric spaces is defined [12]. A subset U of a metric space X is called *large scale doubling* if there is a constant $N \in \mathbb{N}$ such that for every sufficiently large $r > 1$ and for every ball $B_{2r}(x)$ in X of radius $2r$, the intersection $B_{2r}(x) \cap U$ can be covered by at most N balls of radius r.

The hyperbolic dimension of X is the minimal integer $\mathrm{hypdim}(X) = n$ such that for every $d > 0$ the is an open covering $\mathcal{U}$ of X with multiplicity $\leq n+1$ and $L(\mathcal{U}) > d$, which is uniformly large scale doubling.

The following natural question arises.

Question 5.1. Is a counterpart of Theorem 3.1 valid for the hyperbolic dimension hypdim?

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АСИМПТОТИЧНИЙ ВИМІР СИМЕТРИЧНИХ СТЕПЕНІВ

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Основний результат стверджує, що $\operatorname{asdim} SP^n_G X \le n \operatorname{asdim} X$, де X – власний метричний простір, а asdim — асимптотичний вимір у сенсі Громова. Відповідний результат також виконано для однієї модифікації асимптотичного виміру — так званого виміру Ассуада-Нагати. Тут SP^n_G означає функтор G-симетричного степеня.