**FRACTAL CAPACITIES
AND ITERATED FUNCTION SYSTEMS**

©*2008 p. Oleg NYKYFORCHYN*

Vasyl' Stefanyk Precarpathian National University,
Department of Mathematics and Computer Science
Shevchenka 57, Ivano-Frankivsk, Ukraine

Received October 6, 2008

Iterated function systems are defined for inclusion hyperspaces and capacities, and counterparts of classical theorems on attractors, namely the fixed point theorem, the continuity with respect to a contraction, the collage and anti-collage theorems, are proved. Self-similar random capacities are also defined, ad their properties, analogous to properties of random self-similar measures, are investigated.

Introduction

Capacities were introduced by Choquet [1] as a natural generalization of measures. They found numerous applications, e.g. in decision making theory in conditions of uncertainty [2, 3, 4, 5, 6]. Upper semicontinuous capacities were defined and studied in [7]. Algebraic and topological properties of capacities on compact Hausdorff spaced were investigated in [8]. In particular, the capacity functor in the category of compacta was defined. A remarkable fact is that this functor is a functorial part of a monad that is also described in [8]. The aim of this paper is to transfer to capacities remarkable results on fractal measures, in particular, random fractal measures.

MSC 2000: 18B30, 54B30

Дослідження підтримано Державним фондом фундаментальних досліджень України, проект Ф25.1/099.

1 Basic definitions, notations and facts

A *compactum* is a compact Hausdorff topological space. We regard the unit segment $I = [0;1]$ as a subspace of the real line with the natural topology. We write $A \underset{\text{cl}}{\subset} B$ or $A \underset{\text{op}}{\subset} B$ if A is a closed or resp. an open subset of a space B. For a set X the identity mapping $X \to X$ is denoted by $\mathbf{1}_X$.

For a set Y and a metric space (X, d) with $\sup d < \infty$ the uniform convergence metric on the set of all mappings $Y \to X$ is defined by the formula

$$d_u(f, g) = \sup\{d(f(x), g(x)) \mid x \in X\}, \quad f, g : Y \to X.$$

For a topological space X its *hyperspace* $\exp X$ is the set of all closed nonempty subsets of X with the *Vietoris topology*, see, e.g., [9]. The standard base of the latter consists of all sets of the form

$$\langle U_1, \ldots, U_n\rangle = \{F \in \exp X \mid F \subseteq U_1 \cup \cdots \cup U_n, F \cap U_i \neq \varnothing \ \forall\ i = 1, \ldots, n\},$$

where $U_1, \ldots, U_n$ are open sets in X. If (X, d) is a metric compactum, the Vietoris topology on $\exp X$ is determined by the *Hausdorff metric* d_H that is defined as

$$d_H(F, G) = \inf\{\varepsilon \geqslant 0 \mid d(a, B) \leqslant \varepsilon, d(b, A) \leqslant \varepsilon \text{ for all } a \in A, b \in B\},$$

$$F, G \in \exp X,$$

where $d(x, Y) = \inf\{d(x, y) \mid y \in Y\}$ for any $x \in X$, $Y \in \exp X$. It is known that for a compactum X the hyperspace $\exp X$ is a compactum as well, therefore we can consider compacta $\exp^2 X = \exp(\exp X)$, $\exp^3 X = \exp(\exp^2 X)$, etc. For a metric compactum (X, d) the Vietoris topology on $\exp^2 X$ is determined by the metric $d_{HH} = (d_H)_H$, and so forth.

For $\delta \geqslant 0$ and a set A in a metric space (X, d) let $\bar{O}_\delta(A) = \{x \in X \mid d(x, A) \leqslant \delta\}$. In particular, $\bar{O}_\delta(\{a\}) = \bar{B}_\delta(a)$ for $\delta > 0$ is the closed ball with the center a and the radius δ. Then we can equivalently define the Hausdorff metric by the formula

$$d_H(F, G) = \min\{\delta \geqslant 0 \mid F \subset \bar{O}_\delta(G), G \subset \bar{O}_\delta(F)\}.$$

The *diameter* of a set A in a metric space (X, d) is defined to be $\operatorname{diam} A = \sup\{d(x, y) \mid x, y \in A\}$.

An *inclusion hyperspace* $\mathcal{H}$ on a compactum X is a closed subset of $\exp X$ such that $A \in \mathcal{H}$, $A \subset B$ imply $B \in \mathcal{H}$ for all $A, B \in \exp X$ (see [9]). The set GX of all inclusion hyperspaces on X is closed in $\exp^2 X$, therefore GX is a compactum. If (X, d) is a metric compactum, then the topology of GX is determined by the metric d_{HH} .

We follow a terminology of [8] and call a function $c : \exp X \cup \{\varnothing\} \to I$ a *capacity* on a compactum X if the three following properties hold for all closed subsets F, G in X :

(1) $c(\varnothing) = 0$, $c(X) = 1$;

(2) if $F \subseteq G$, then $c(F) \leqslant c(G)$ (monotonicity);

(3) if $c(F) < a$, then there exists an open set $U \supseteq F$ such that for any $G \subseteq U$ we have $c(G) < a$ (upper semicontinuity).

We extend a capacity c to all open subsets in X by the formula :

$$c(U) = \sup\{c(F) \mid F \underset{\mathrm{cl}}{\subset} X, F \subseteq U\}.$$

It is proved in [8] that the set MX of all capacities on a compactum X is a compactum as well, if a topology on MX is determined by a subbase that consists of all sets of the form

$$O_-(F, a) = \{c \in MX \mid c(F) < a\},$$

where $F \underset{\mathrm{cl}}{\subset} X$, $a \in \mathbb{R}$, and

$$O_+(U, a) = \{c \in MX \mid c(U) > a\} = \\ \{c \in MX \mid \text{there exists a compactum } F \subseteq U, c(F) > a\},$$

where $U \underset{\mathrm{op}}{\subset} X$, $a \in \mathbb{R}$.

If the topology on a compactum X is determined by a compatible metric d, then [8] the topology on MX is determined by the following metric :

$$\hat{d}(c, c') = \inf\{\varepsilon > 0 \mid \forall F \underset{\mathrm{cl}}{\subset} X \; c(\bar{O}_\varepsilon(F)) + \varepsilon \geqslant c'(F), c'(\bar{O}_\varepsilon(F)) + \varepsilon \geqslant c(F)\}.$$

We write $c_1 \leqslant c_2$ for $c_1, c_2 \in MX$ iff $c_1(F) \leqslant c_2(F)$ for all $F \underset{\mathrm{cl}}{\subset} X$. Then MX is a Lawson lattice [10], and for any set $\{c_i \in MX | i \in \mathcal{I}\}$ and $F \underset{\mathrm{cl}}{\subset} X$ we have $\bigvee_{i \in \mathcal{I}} c_i(F) = \sup\{c_i(F) \mid i \in \mathcal{I}\}$, $\bigwedge_{i \in \mathcal{I}} c_i(F) = \inf\{c_i(F) \mid i \in \mathcal{I}\}$.

The assignments exp, G and M extend respectively to the *hyperspace functor*, *inclusion hyperspace functor* and *capacity functor* with the same denotations in the category of compacta, if the maps $\exp f : \exp X \to \exp Y$,

$Gf : GX \to GY$ and $Mf : MX \to MY$ for a continuous map of compacta $f : X \to Y$ are defined by the formulae

$$\exp f(F) = \{f(x) \mid x \in F\}, \quad F \in \exp X,$$

$$Gf(\mathcal{H}) = \{B \underset{\mathrm{cl}}{\subset} Y \mid B \supset f(A) \text{ for some } A \in \mathcal{H}\}, \quad \mathcal{H} \in GX,$$

$$Mf(c)(F) = c(f^{-1}(F)), \quad c \in MX, F \underset{\mathrm{cl}}{\subset} Y.$$

We will also use the mapping $\mu X : M^2X \to MX$ defined in [8] by the formula

$$\mu X(\mathcal{C})(F) = \sup\{\ \alpha \in I \mid \mathcal{C}(\{c \in MX \mid c(F) \geqslant \alpha\}) \geqslant \alpha\},$$

where $\mathcal{C} \in M^2X$, $F \underset{\mathrm{cl}}{\subset} X$. It is the component of the multiplication of the capacity monad, see [8] for algebraic meaning of this mapping and [10] for a "practical interpretation". We will use only the fact that μX is continuous. In the sequel we denote the set $\{c \in MX \mid c(F) \geqslant \alpha\}$ by F_α. Sometimes it is more convenient to use an equivalent definition of μX : $\mu X(\mathcal{C})(F) \geqslant \alpha$ for $\mathcal{C} \in M^2X$, $F \underset{\mathrm{cl}}{\subset} X$, $\alpha \in I$ iff there is a set $\mathcal{F} \underset{\mathrm{cl}}{\subset} MX$ such that $\mathcal{C}(\mathcal{F}) \geqslant \alpha$, and $c(F) \geqslant \alpha$ for all $c \in \mathcal{F}$.

For $c \in MX$ and $\alpha \in I$ the *α-section* of c is the set $S_\alpha c = \{F \in \exp X \mid c(F) \geqslant \alpha\}$. It is proved in [8] that $S_\alpha c \in GX$, and the collection of all $S_\alpha c$, $\alpha \in I$, uniquely determines a capacity c. The *subgraph* of a capacity $c \in MX$ is a set $\operatorname{sub} c = \{(F, \alpha) \in \exp X \times I \mid c(F) \geqslant \alpha\}$. It is proved in [10] that $\operatorname{sub} c$ is closed in $\exp X \times I$, and the mapping $\operatorname{sub} : MX \to \exp(\exp X \times I)$ is an embedding. Obviously $\operatorname{sub} c \cap (\exp X \times \{\alpha\}) = S_\alpha c \times \{\alpha\}$.

We will use a

Lemma 1. *Let X be a compact metric space and a metric $\bar{d}$ on $\exp X \times I$ is defined by the formula $\bar{d}((F_1, \alpha_1), (F_2, \alpha_2)) = \max\{d_H(F_1, F_2), |\alpha_1 - \alpha_2|\}$, where $F_1, F_2 \in \exp X$, $\alpha_1, \alpha_2 \in I$. Then for all $c_1, c_2 \in MX$ the equality $\hat{d}(c_1, c_2) = \bar{d}_H(\operatorname{sub} c_1, \operatorname{sub} c_2)$ holds.*

Proof is straightforward.

We call a capacity $c \in MX$ a $\cup$-*capacity* (also called *sup-measure* or *possibility measure*, [11]), if $c(A \cup B) = \max\{c(A), c(B)\}$ for all $A, B \underset{\mathrm{cl}}{\subset} X$. Each $\cup$-capacity c is completely determined by its values on singletons : $c(A) = \max\{c(\{x\}) \mid x \in A\}$ for a set $A \subset X$, therefore we identify c with the upper semicontinuous function $X \to I$ that sends each x to $c(\{x\})$. We preserve the same denotation c for this function. The set $M_\cup X$ is closed in MX, and for a continuous mapping $f : X \to Y$ of compacta we have $Mf(M_\cup X) \subset M_\cup Y$, so we can define $M_\cup f : M_\cup X \to M_\cup Y$ as a restriction of Mf. Thus a *subfunctor* $M_\cup$ of the functor M in the category of compacta is determined [9, 11]. Moreover, $\mu X(M_\cup^2 X) \subset M_\cup X$, and we define $\mu_\cup X : M_\cup^2 X \to M_\cup X$ as a restriction of μX. If $\cup$-capacities on a compactum Y are regarded as functions $Y \to I$, then $\mu_\cup X$ is determined by the formula :

$$\mu_\cup X(\mathcal{C})(x) = \sup\{\alpha \in I \mid \exists\, c \in M_\cup X \text{ such that } \mathcal{C}(c) \geqslant \alpha, c(x) \geqslant \alpha\},$$

where $\mathcal{C} \in M_\cup^2 X$, $x \in X$.

2 Main results

In the sequel let X be a compact metric space. For a mapping $f : X \to X$ the *contraction factor* is defined to be equal to

$$\operatorname{Lip} f = \sup\{\frac{d(f(x), d(y))}{d(x,y)} \mid x, y \in X, x \neq y\}.$$

A mapping f such that $\operatorname{Lip} f < 1$ is called a *contraction*, and f is *non-expanding* if $\operatorname{Lip} f \leqslant 1$. For $0 < q < 1$ we denote $R_q(X) = \{r : X \to X \mid \operatorname{Lip} r \leqslant q\}$. It is easy to see that $R_q(X)$ is a compactum with the uniform convergence metric.

Recall how a classical *iterated function system* (IFS) for sets is defined. Usually only *finite* sets of contractions are involved because of their practical use, but there is no formal need for such a restriction. Thus in the sequel IFS $\bar{r}$ is a *closed nonempty* set of contractions with contraction factors not greater than some $q < 1$, i.e. $\bar{r} \in \exp R_q(X)$. Then for any $F \in \exp X$ we put $\exp \bar{r}(F) = \bigcup_{r \in \bar{r}} \exp r(F)$. It is well-known that the mapping $\exp \bar{r}$ is a contraction in the space $\exp X$ with the Hausdorff metric, and $\operatorname{Lip} \exp \bar{r} \leqslant q$. Thus it is possible to apply to $\exp \bar{r}$ four classical theorems on contractions :

Theorem (Banach fixed point theorem for contraction maps, [12])**.** *Let (Y, d) be a complete metric space and $f : Y \to Y$ be a mapping such that $\operatorname{Lip} f \leqslant q < 1$. Then there is a unique $y_0 \in Y$ such that $f(y_0) = y_0$. Moreover, for any $y \in Y$ and $n \in \mathbb{N}$, $d(f^n(y), y_0) \leqslant \frac{q^n \operatorname{diam} Y}{1-q}$, thus $f^n(y) \to y_0$ as $n \to \infty$.*

Theorem (Continuity of fixed points with respect to contraction maps, [13])**.** *Let (Y, d) be a compact metric space and contractions $f, g : Y \to Y$ have fixed points y_{0f} and y_{0g} respectively. Then*

$$d(y_{0f}, y_{0g}) \leqslant \frac{d_u(f,g)}{1 - \min\{\operatorname{Lip} f, \operatorname{Lip} g\}}.$$

Theorem ("Collage theorem", [14]). *Let (Y, d) be a complete metric space and f be a contraction with a fixed point y_0. Then for any $y \in Y$,*

$$d(y, y_0) \leqslant \frac{1}{1 - \operatorname{Lip} f} d(y, f(y)).$$

Theorem ("Anti-Collage theorem", [15]). *Assume the conditions of the previous theorem. Then for any $y \in Y$,*

$$d(y, y_0) \geqslant \frac{1}{1 + \operatorname{Lip} f} d(y, f(y)).$$

Since $\exp X$ is complete, there is a unique fixed point F for $\exp \bar{r}$, i.e. a set F such that $\exp \bar{r}(F) = F$. This fixed point is called the *attractor* of the IFS $\bar{r}$ or a *fractal set self-similar w.r.t.* $\exp \bar{r}$. For any $H \in \exp X$ the sequence $(\exp \bar{r})^n(F)$, $n = 1, 2, \dots$, converges to the fixed point exponentially fast.

Now we extend this notion to inclusion hyperspaces. For any $\bar{r} \in \exp R_q(X)$ and $\mathcal{F} \in GX$ we put $G\bar{r}(c) = \bigcap_{r \in \bar{r}} Gr(\mathcal{F})$. Then $G\bar{r}(\mathcal{F})$ is in GX and depends continuously on $(\bar{r}, c) \in \exp R_q(X) \times GX$. Now for $\mathcal{R} \in MR_q(X)$ we define $G\mathcal{R}(\mathcal{F})$ by the formula $G\mathcal{R}(\mathcal{F}) = \bigcup_{\bar{r} \in \mathcal{R}} G\bar{r}(\mathcal{F})$. It is easy to observe that for $H \in \exp X$ we have $H \in G\mathcal{R}(\mathcal{F})$ if and only if there is $\bar{r} \underset{\mathrm{cl}}{\subset} R_q(X)$, $\bar{r} \in \mathcal{R}$ such that for each $r \in \bar{r}$ the set H contains the image $r(F)$ of some $F \in \mathcal{F}$. It is straightforward to check that $G\mathcal{R}(\mathcal{F}) \in GX$ and it depends continuously on $(\mathcal{F}, \mathcal{R}) \in GX \times MR_q(X)$. It differs from the usual IFS for compact sets in that each contraction has a "choice" on which set to act in a given inclusion hyperspace. Following the commonly used terminology style (see, e.g. [16]) we call $\mathcal{R}$ an *IFS for inclusion hyperspaces* and $G\mathcal{R}$ an *IFS operator* or *fractal transform* associated with $\mathcal{R}$. The functors exp and G preserve contraction factors of mappings, thus

Theorem 1. *If $\mathcal{R} \in GR_q(X)$, then $G\mathcal{R} \in R_q(GX)$.*

(An obvious proof is omitted.) Therefore the four previous theorems about contractions are applicable to $G\mathcal{R}$ too. Thus a fixed point $\mathcal{F}$ for $G\mathcal{R}$ exists, is unique and depends continuously on $\mathcal{R}$. It is natural to call it the *attractor* of the IFS $\mathcal{R}$ or a *fractal inclusion hyperspace self-similar w.r.t.* $G\mathcal{R}$.

We will use the two (of many existing) natural embeddings $i_G X, i^G X : \exp X \hookrightarrow GX$ for a compactum X, namely $i_G X(F) = \{H \in \exp X \mid H \supset F\}$, $i^G X(F) = \{H \in \exp X \mid H \cap F \neq \varnothing\}$ for $F \in \exp X$. For a fixed $\bar{r} \in \exp R_q(X)$ let $\mathcal{R}_* = i_G R_q(X)(\bar{r})$, $\mathcal{R}^* = i^G R_q(X)(\bar{r})$. Then it is easy to verify that for any $F \in \exp X$ we have $G\mathcal{R}_*(i_G X(F)) = i_G X(G\bar{r}(F))$,

$G\mathcal{R}^*(i^G X(F)) = i^G X(G\bar{r}(F))$. Thus a fractal transform for sets embeds into a fractal transform for inclusion hyperspaces.

As inclusion hyperspaces are tightly connected with capacities ([8]), it is natural to go forth and define IFS for capacities. As MX is a Lawson lattice w.r.t. "setwise" infs and sups, we can put $M\bar{r}(c) = \bigwedge_{r\in\bar{r}} Mr(c)$ for all $\bar{r} \in \exp R_q(X)$ and $c \in MX$. Then $M\bar{r}(c)$ is in MX and depends continuously on $(\bar{r}, c) \in \exp R_q(X) \times MX$. Now for $\mathcal{R} \in MR_q(X)$ we define $M\mathcal{R}(c)$ by the formula

$$M\mathcal{R}(c)(F) = \bigvee_{\bar{r}\in\exp R_q(X)} \min\{M\bar{r}(c)(F), \mathcal{R}(\bar{r})\} \text{ for } F \underset{\text{cl}}{\subset} X.$$

Theorem 2 (Fixed point theorem for capacities)**.** *Let X be a metric compactum, $c \in MX$ and $\mathcal{R} \in MR_q(X)$. Then $M\mathcal{R}(c)$ is a capacity on X, and the mapping $M\mathcal{R}$ is non-expanding but is not a contraction. Nevertheless, there is a unique $c_0 \in MX$ such that $M\mathcal{R}(c_0) = c_0$, and for any $c \in MX$ we have $\hat{d}((M\mathcal{R})^n(c), c_0) \leqslant q^n \operatorname{diam} X$, thus $(M\mathcal{R})^n(c) \to c_0$ as $n \to \infty$.*

Proof. It is obvious that $M\mathcal{R}(c)(\varnothing) = 0$, $M\mathcal{R}(c)(X) = 1$, and $A \subset B$, $A, B \underset{\text{cl}}{\subset} X$ imply $M\mathcal{R}(c)(A) \leqslant M\mathcal{R}(c)(B)$. If $M\mathcal{R}(c)(F) < \alpha \in I$, then there is no such $\bar{r} \in \exp R_q(X)$ that $\mathcal{R}(\bar{r}) \geqslant \alpha$ and $Mr(c)(F) \geqslant \alpha$ for all $r \in \bar{r}$. Therefore the capacity $\mathcal{R}$ of the closed set $\{r \in R_q(X) \mid Mr(c) \in F_\alpha\}$ is less than α. It was proved in [17] that F_α depends continuously on (F, α), thus there is a neighborhood $U \supset F$ in X such that for any $H \in \exp X$, $H \subset U$ we also have $\mathcal{R}(\{r \in R_q(X) \mid Mr(c) \in H_\alpha\}) < \alpha$, which implies $M\mathcal{R}(c)(H) \leqslant \alpha$. This is sufficient for the upper semicontinuity of $M\mathcal{R}(c)$, and this function is a capacity.

To prove that $M\mathcal{R}$ is non-expanding, we first observe that for a non-expanding $r : X \to X$ the mapping $Mr : MX \to MX$ is non-expanding. Next, if $(c_i)_{i\in\mathcal{I}}$ and $(c'_i)_{i\in\mathcal{I}}$ are collections of capacities on X such that $\hat{d}(c_i, c'_i) \leqslant \varepsilon$ for all $i \in \mathcal{I}$, then $\hat{d}(\bigwedge_{i\in\mathcal{I}} c_i, \bigwedge_{i\in\mathcal{I}} c'_i) \leqslant \varepsilon$, therefore for all $\bar{r} \in \exp R_q(X)$ and $c, c' \in MX$ the inequality $\hat{d}(M\bar{r}(c), M\bar{r}(c')) = \hat{d}(\bigwedge_{r\in\bar{r}} Mr(c), \bigwedge_{r\in\bar{r}} Mr(c')) \leqslant \hat{d}(c, c')$ holds, which implies $\bar{d}_H(\operatorname{sub} M\bar{r}(c), \operatorname{sub} M\bar{r}(c')) \leqslant \hat{d}(c, c')$ by Lemma 1. If a mapping $\varphi_\beta : I \to I$ for $\beta \in I$ is defined as $\varphi_\beta(t) = \min\{t, \beta\}$, then the mapping $\mathbf{1}_{\exp X} \times \varphi_\beta : \exp X \times I \to \exp X \times I$ is also non-expanding w.r.t. the metric $\bar{d}$ defined in Lemma 1. As the operation of union in a metric compactum Y is also non-expanding as mapping $\exp^2 Y \to \exp Y$, we obtain that the mapping that sends $c \in MX$ to

$$\operatorname{sub} M\mathcal{R}(c) = \bigcup_{\bar{r} \in \exp R_q(X)} \exp(\mathbf{1}_{\exp X} \times \varphi_{\mathcal{R}(\bar{r})})(\operatorname{sub} M\bar{r}(c))$$

is non-expanding. By Lemma 1 this means that $M\mathcal{R}$ is non-expanding.

Due to size restrictions we omit a simple example of $c_1, c_2 \in MX$ such that

$$\hat{d}(M\mathcal{R}(c_1), M\mathcal{R}(c_1)) = \hat{d}(c_1, c_2) \neq 0.$$

Let us study the section $S_\alpha M\mathcal{R}(c) = \{F \in \exp X \mid M\mathcal{R}(c) \geqslant \alpha\} \in GX$. Then $F \in S_\alpha M\mathcal{R}(c)$ iff there is $\bar{r} \in S_\alpha \mathcal{R}$ such that $F \in \bigcap_{r \in \bar{r}} S_\alpha Mr(c) = \bigcap_{r \in \bar{r}} Gr(S_\alpha c)$. Thus $S_\alpha M\mathcal{R}(c) = \bigcup_{\bar{r} \in S_\alpha} \bigcap_{r \in \bar{r}} Gr(S_\alpha c) = G(S_\alpha \mathcal{R})(S_\alpha c)$, and IFS $\mathcal{R}$ for capacities acts on each section $S_\alpha c$ as the IFS $S_\alpha \mathcal{R}$ for inclusion hyperspaces. As for inclusion hyperspaces fixed points for IFSs are unique, a fixed point $c_{\mathcal{R}}$ for $M\mathcal{R}$ is unique as well.

Observe that if a metric compactum (Y, d) is a union of its closed subsets Y_i, and closed subsets $A, B \subset Y$ intersect all Y_i, then $d_H(A, B) \leqslant \sup_i \{d_H(A \cap Y_i, B \cap Y_i)\}$. Thus for $c_1, c_2 \in MX$ we obtain

$$\begin{aligned} \hat{d}(c_1, c_2) &= \bar{d}_H(\operatorname{sub} c_1, \operatorname{sub} c_2) \leqslant \\ &\leqslant \sup_{\alpha \in I} \bar{d}_H(\operatorname{sub} c_1 \cap (\exp X \times \{\alpha\}), \operatorname{sub} c_2 \cap (\exp X \times \{\alpha\})) = \\ &= \sup_{\alpha \in I} d_{HH}(S_\alpha c_1, S_\alpha c_2). \end{aligned}$$

The right side of the latter inequality is also a metric on the space MX ([18]). Let us denote it $d_\infty(c_1, c_2)$. By Theorem 1 the mapping $M\mathcal{R}$ is a contraction with a factor $\leqslant q$ w.r.t. the metric d_∞. As $\sup d_\infty = \operatorname{diam} X$, by the above we obtain that $\hat{d}((M\mathcal{R})^{n-1}(c), (M\mathcal{R})^n(c)) \leqslant d_\infty((M\mathcal{R})^{n-1}(c), (M\mathcal{R})^n(c) \leqslant q^{n-1} \operatorname{diam} X$ for all $c \in MX$, $n \in \mathbb{N}$. Thus the sequence $(M\mathcal{R})^n(c)$ converges to some $c_0 \in MX$, and by continuity of $M\mathcal{R}$ the capacity c_0 is a fixed point. Similarly $(M\mathcal{R})^n$ is a contraction with a factor $\leqslant q^n$ w.r.t. the metric d_∞, thus for any $c, c' \in MX$ we have $\hat{d}((M\mathcal{R})^n(c), (M\mathcal{R})^n(c')) \leqslant q^n \operatorname{diam} X$, thus

$$\hat{d}((M\mathcal{R})^n(c), c_0) = \hat{d}((M\mathcal{R})^n(c), (M\mathcal{R})^n(c_0)) \leqslant q^n \operatorname{diam} X.$$

□

We call $M\mathcal{R}$ a *scaling law for capacities* (following [19]) of *fractal transform for capacities* (like [20]), and $\mathcal{R}$ is an *IFS for capacities.* If $c = M\mathcal{K}(c)$, then c is an *attractor* of $\mathcal{R}$ or a *capacity that is self-similar w.r.t. $M\mathcal{R}$.*

Lemma 2. *For a fixed $c \in MX$ the mapping $(MR_q(X), \hat{d}_u) \to (MX, \hat{d})$, that sends $\mathcal{R}$ to $M\mathcal{R}(c)$, is non-expanding.*

Proof. If $r, r' \in R_q(X)$ are such that $d_u(r, r') \leqslant \delta$, then obviously $\hat{d}(Mr(c), Mr'(c)) \leqslant \delta$. If $\mathcal{A}, \mathcal{B} \in \exp MX$ are such that $\hat{d}_H(\mathcal{A}, \mathcal{B}) \leqslant \delta$, then $\hat{d}(\vee\mathcal{A}, \vee\mathcal{B}) \leqslant \delta$, $\hat{d}(\wedge\mathcal{A}, \wedge\mathcal{B}) \leqslant \delta$. Combining these two facts together, we obtain that if $\bar{r}, \bar{r}' \in \exp R_q(X)$, $(d_u)_H(\bar{r}, \bar{r}') \leqslant \delta$, $c \in MX$, then

$$\hat{d}(M\bar{r}(c), M\bar{r}'(c)) = \hat{d}(\bigwedge_{r \in \bar{r}} Mr(c), \bigwedge_{r \in \bar{r}'} Mr(c)) \leqslant \delta.$$

Now let $\hat{d}_u(\mathcal{R}, \mathcal{R}') = \delta$ for $\mathcal{R}, \mathcal{R}' \in MR_q(X)$. For a set $F \in \exp X$ and $c \in MX$ we denote $M\mathcal{R}(c)(F) = \alpha$. Then there exists $\bar{r} \in \exp R_q(X)$ such that $\mathcal{R}(\bar{r}) \geqslant \alpha$, $M\bar{r}(c)(F) \geqslant \alpha$. Put $\bar{r}' = \bar{O}_\delta(\bar{r})$, then $\mathcal{R}'(\bar{r}') \geqslant \alpha - \delta$, $(d_u)_H(\bar{r}, \bar{r}') \leqslant \delta$, therefore $\hat{d}(M\bar{r}(c), M\bar{r}'(c)) \leqslant \delta$. Thus $M\bar{r}'(c)(\bar{O}_\delta(F)) \geqslant M\bar{r}(c)(F) - \delta \geqslant \alpha - \delta$, and $M\mathcal{R}'(c)(\bar{O}_\delta(F)) \geqslant \alpha - \delta = M\mathcal{R}(c)(F) - \delta$, i.e. $M\mathcal{R}(c)(F) \leqslant M\mathcal{R}'(c)(\bar{O}_\delta(F)) + \delta$. Similarly we prove $M\mathcal{R}'(c)(F) \leqslant M\mathcal{R}(c)(\bar{O}_\delta(F)) + \delta$ for all $F \in \exp X$. This implies $\hat{d}(M\mathcal{R}(c), M\mathcal{R}'(c)) \leqslant \delta = \hat{d}_u(\mathcal{R}, \mathcal{R}')$. □

Theorem 3 (Continuity of fixed points with respect to IFS)**.** *Let (X, d) be a metric compactum, and let c_0, c_0' be attractors for $\mathcal{R}, \mathcal{R}' \in MR_q(X)$ respectively. Then $\hat{d}(c_0, c_0') \leqslant \sum_{n=1}^{\infty} \min\{\hat{d}_u(\mathcal{R}, \mathcal{R}'), 2q^{n-1} \operatorname{diam} X\}$, therefore $\hat{d}(c_0, c_0') \to 0$ as $\hat{d}_u(\mathcal{R}, \mathcal{R}') \to 0$.*

Proof. We denote $\hat{d}_u(\mathcal{R}, \mathcal{R}') = \delta$. By the above for a capacity $c \in MX$ we have $\hat{d}(M\mathcal{R}(c), M\mathcal{R}'(c)) \leqslant \delta$, $d((M\mathcal{R})^2(c), (M\mathcal{R}')^2(c)) \leqslant 2\delta, \ldots$, $d((M\mathcal{R})^n(c), (M\mathcal{R}')^n(c)) \leqslant n\delta, \ldots$. Let n_0 be a least index n such that $\delta > 2q^{n-1}$. Then

$$\hat{d}(c_0, c_0') \leqslant$$

$$\leqslant \hat{d}(c_0, (M\mathcal{R})^{n_0-1}(c)) + d((M\mathcal{R})^{n_0-1}(c), (M\mathcal{R}')^{n_0-1}(c)) +$$

$$+ d((M\mathcal{R}')^{n_0-1}(c), c_0') \leqslant \sum_{n=n_0}^{\infty} \hat{d}((M\mathcal{R})^{n-1}(c), (M\mathcal{R})^n(c)) +$$

$$+ (n_0 - 1)q + \sum_{n=n_0}^{\infty} \hat{d}((M\mathcal{R}')^{n-1}(c), (M\mathcal{R}')^n(c)) \leqslant$$

$$\leqslant \sum_{n=n_0}^{\infty} q^{n-1} \operatorname{diam} X + (n_0 - 1)q + \sum_{n=n_0}^{\infty} q^{n-1} \operatorname{diam} X =$$

$$= \sum_{n=1}^{\infty} \min\{\delta, 2q^{n-1} \operatorname{diam} X\}.$$

□

In a quite similar manner we obtain a

Theorem 4 ("Collage+Anti-Collage theorem"for capacities)**.** *Let X be a metric compactum, $c \in MX$ and $\mathcal{R} \in MR_q(X)$. If c_0 is a fixed point of $M\mathcal{R}$, then*

$$\frac{1}{2}\hat{d}(c, M\mathcal{R}(c)) \leqslant \hat{d}(c, c_0) \leqslant \sum_{n=1}^{\infty} \min\{\hat{d}(c, M\mathcal{R}(c)), q^{n-1} \operatorname{diam} X\}.$$

This theorem provides a ground for solutions of the *inverse problem for capacities* : given $c \in MX$ and a class $\mathcal{M} \subset MR_q(X)$ of IFSs, find $\mathcal{R} \in \mathcal{M}$ such that the attractor c_0 of $\mathcal{R}$ is close enough to c (see [16]).

Now we show that the proposed transform includes a simple variant of the method of Iterated Fuzzy Sets Systems (IFZS, see [21]). Each $\cup$-capacity $c \in M_\cup X$ can be treated as an upper-continuous function $X \to I$, that is a fuzzy subset of X with compact level sets. If c is considered as an image in X, then for any point x the value $c(x)$ is a *grey level* (0 = black, 1 = white).

Assume that $\mathcal{R} \in M_\cup R_q(X)$ and look how $M\mathcal{R}$ acts on $c \in M_\cup X$. It is straightforward to verify that $M\mathcal{R}(c) \in M_\cup X$, and

$$\begin{aligned} M\mathcal{R}(x) = \sup\{\alpha \in I \mid \text{there are } r \in R_q(X), y \in X \text{ such that } \mathcal{R}(r) \geqslant \alpha, \\ c(y) \geqslant \alpha, r(y) = x\} = \sup\{\varphi_{\mathcal{R}(r)}(c(y)) \mid r \in R_q(X), y \in r^{-1}(x)\}, \end{aligned}$$

where again $\varphi_\beta(t) = \min\{\beta, t\}$ for $t \in I$. It means that we make transformed copies of the image c, but restrict brightness of the copy of c under r from the above by $\mathcal{R}(r)$. If $\mathcal{R}(r) \neq 0$ only for a finite number of $r \in R_q(X)$, then we obtain IFZS, and $\varphi_{\mathcal{R}(r)}$, $r \in R_q(X)$, are simple *grey level maps* [16].

All the described above fractal transforms were deterministic, i.e. they transform each inclusion hyperspace or capacity into a uniquely determined object. Now we will study how is it possible to obtain *random* fractal capacities. It is natural to exploit the fact that capacities are a natural framework to reflect uncertainty. If X is considered as a space of elementary events (sample space) for some experiment, and c is a capacity on X, then $c(A)$ for a subset $A \subset X$ is a level of certainty that some event $x \in A$ will appear in the experiment. The more is the value $c(A) \in [0; 1]$, the more probable we consider the event A. We can say that c describes a capacity distribution of a random point $x \in X$.

Now we describe a transform that is a counterpart of the scaling law for random measures defined by Hutchinson, Rüschendorf in [19] and of superfractals introduced by Barnsley, Hutchinson and Stenflo [22, 23].

For a fixed $\mathcal{C} \in M^2X$ we define a mapping $\psi_{\mathcal{C}} : MR_q(X) \to M^2X$ by the formula $\psi_{\mathcal{C}}(\mathcal{R}) = M(M\mathcal{R})(\mathcal{C})$. By Lemma 2 the mapping $\psi_{\mathcal{C}}$ is nonexpanding, and by Theorem 2 the mapping that sends each $\mathcal{C} \in M^2X$ to $\psi_{\mathcal{C}}$, is nonexpanding as well w.r.t. the pair of the metric $\hat{\hat{d}}$ and the uniform convergence metric. Now we fix a "big coefficient" $\mathcal{K} \in M^2R_q(X)$. It describes a capacity distribution of a "small coefficient" $\mathcal{R} \in MR_q(X)$. The functor M preserves the class of nonexpanding mappings, therefore the mapping $M\psi_{\mathcal{C}} : M^2R_q(X) \to M^3X$ is nonexpanding, as well as the mapping $\mathcal{C} \mapsto M\psi_{\mathcal{C}}$. We put $\Psi_{\mathcal{K}}(\mathcal{C}) = \mu MX \circ M\psi_{\mathcal{C}}(\mathcal{K})$. Taking into account that μY is nonexpanding for any metric compactum Y, we conclude that the mapping $\Psi_{\mathcal{K}} : M^2X \to M^2X$ is nonexpanding. It is not a contraction, therefore usual contraction arguments are not directly applicable here to prove the existence and the uniqueness of a fixed point for $\Psi_{\mathcal{K}}$. We are to examine properties of $\Psi_{\mathcal{K}}$ deeper.

Lemma 3. *Let* $\mathcal{K} \in M^2R_q(X)$ *and* $\mathcal{C}, \mathcal{C}' \in M^2X$. *Then* $\hat{\hat{d}}((\Psi_{\mathcal{K}})^n(\mathcal{C}), (\Psi_{\mathcal{K}})^n(\mathcal{C}')) \leqslant q^n \operatorname{diam} X$.

Proof. We denote $\mathrm{C} = M\psi_{\mathcal{C}}(\mathcal{K})$. Then $\Psi_{\mathcal{K}}(\mathcal{C})(\mathcal{F}) \geqslant \alpha$ for $\mathcal{F} \underset{\mathrm{cl}}{\subset} MX$, $\alpha \in I$ if and only if there exists $\mathcal{H} \underset{\mathrm{cl}}{\subset} M^2X$ such that $\mathrm{C}(\mathcal{H}) \geqslant \alpha$, and for all $C' \in \mathcal{H}$ we have $C'(\mathcal{F}) \geqslant \alpha$. This is equivalent to the existence of $H \underset{\mathrm{cl}}{\subset} MR_q(X)$ such that $\mathcal{K}(H) \geqslant \alpha$, and for all $\mathcal{R} \in H$ we have $\psi_{\mathcal{C}}(\mathcal{R})(\mathcal{F}) \geqslant \alpha$, i.e. $M(M\mathcal{R})(\mathcal{C})(\mathcal{F}) \geqslant \alpha$. Thus for $\mathcal{F} \underset{\mathrm{cl}}{\subset} MX$ we have $\mathcal{F} \in S_\alpha \Psi_{\mathcal{K}}(\mathcal{C})$ iff there is $H \in S_\alpha \mathcal{K}$ such that for any $\mathcal{R} \in H$ there is $F \in S_\alpha \mathcal{C}$ such that $M\mathcal{R}(F) \subset \mathcal{F}$. Therefore $\mathcal{F} \in S_\alpha \Psi_{\mathcal{K}} \Psi_{\mathcal{K}}(\mathcal{C})$ iff there is $H \in S_\alpha \mathcal{K}$ such that for all $\mathcal{R} \in H$ there is $H_{\mathcal{R}} \in S_\alpha \mathcal{K}$ such that for all $\mathcal{R}' \in H_{\mathcal{R}}$ there is $F \in S_\alpha \mathcal{C}$ such that $M\mathcal{R} \circ M\mathcal{R}'(F) \subset \mathcal{F}$.

To proceed, for an inclusion hyperspace $G \in GY$ and $n \in \mathbb{N}$ we define an n-level G-tree in the following manner : $\mathcal{H} \subset G \times Y \times G \times \dots \times Y \times G \times Y$ ($2n$ factors) is an n-level G-tree iff the following holds :

1) If $(A_1, x_1, A_2, x_2, \dots, x_{n-1}, A_n, x_n) \in \mathcal{H}$, then $x_1 \in A_1$, $x_2 \in A_2$, $\dots$, $x_{n-1} \in A_{n-1}$, $x_n \in A_n$;

2) For any $x_1 \in A_1 \in G$, $x_2 \in A_2 \in G$, $\dots$, $x_k \in A_k \in G$, $k \in \{0, 1, \dots, n-1\}$ there is a unique $A_{k+1} \in G$ there is $(A_1, x_1, A_2, x_2, \dots, A_{k+1}, \dots, x_{n-1}, A_n, x_n) \in \mathcal{H}$.

The latter property means that A_{k+1} is completely determined by $x_1,\dots,x_k$, therefore for a tree $\mathcal{H}$ in the sequel we denote $A_{k+1}=H_{x_1x_2\dots x_k}$ (thus $A_1=H$ is unique for a fixed tree $\mathcal{H}$).

Now it is straightforward to verify that for $\mathcal{F}\underset{\text{cl}}{\subset} MX$ we have $\mathcal{F}\in S_\alpha(\Psi_{\mathcal{K}})^n(\mathcal{C})$ iff there is an n -level $S_\alpha\mathcal{K}$-tree $\mathcal{H}$ such that for all

$$(H,\mathcal{R}_1,H_{\mathcal{R}_1},\mathcal{R}_2,H_{\mathcal{R}_1\mathcal{R}_2},\dots,\mathcal{R}_{n-1},H_{\mathcal{R}_1\mathcal{R}_2\dots\mathcal{R}_{n-1}},H_n,\mathcal{R}_n)\in\mathcal{H}$$

there is $F\in S_\alpha\mathcal{C}$ such that $M\mathcal{R}_1\circ M\mathcal{R}_2\circ\dots\circ M\mathcal{R}_n(F)\subset\mathcal{F}$.

The mapping $M\mathcal{R}_1\circ M\mathcal{R}_2\circ\dots\circ M\mathcal{R}_n: MX\to MX$ is a contraction with factor $\leqslant q^n$ w.r.t. the metric d_∞ (see proof of Theorem 2). As $\hat{d}\leqslant d_\infty$, we obtain $\operatorname{diam}(M\mathcal{R}_1\circ M\mathcal{R}_2\circ\dots\circ M\mathcal{R}_n(MX))\leqslant q^n\operatorname{diam}X$. Thus $\hat{d}_H(M\mathcal{R}_1\circ M\mathcal{R}_2\circ\dots\circ M\mathcal{R}_n(F),M\mathcal{R}_1\circ M\mathcal{R}_2\circ\dots\circ M\mathcal{R}_n(F'))$ for all $F,F'\underset{\text{cl}}{\subset} MX$. This implies than for any n-level $S_\alpha\mathcal{K}$-tree $\mathcal{H}$ and any collections of $F_{\mathcal{R}_1\mathcal{R}_2\dots\mathcal{R}_{n-1}\mathcal{R}_n},F'_{\mathcal{R}_1\mathcal{R}_2\dots\mathcal{R}_{n-1}\mathcal{R}_n}\subset MX$ for all

$$(H,\mathcal{R}_1,H_{\mathcal{R}_1},\mathcal{R}_2,H_{\mathcal{R}_1\mathcal{R}_2},\dots,\mathcal{R}_{n-1},H_{\mathcal{R}_1\mathcal{R}_2\dots\mathcal{R}_{n-1}},\mathcal{R}_n)\in\mathcal{H}$$

we have

$$\begin{aligned}\hat{d}_H(&\operatorname{Cl}(\bigcup_{(H,\mathcal{R}_1,\dots,\mathcal{R}_n)\in\mathcal{H}} M\mathcal{R}_1\circ M\mathcal{R}_2\circ\dots\circ M\mathcal{R}_n(F_{\mathcal{R}_1\dots\mathcal{R}_n})),\\ &\operatorname{Cl}(\bigcup_{(H,\mathcal{R}_1,\dots,\mathcal{R}_n)\in\mathcal{H}} M\mathcal{R}_1\circ M\mathcal{R}_2\circ\dots\circ M\mathcal{R}_n(F'_{\mathcal{R}_1\dots\mathcal{R}_n})))\leqslant q^n\operatorname{diam}X.\end{aligned}$$

Therefore for all $\alpha\in I$ the distance $\hat{d}_{HH}$ between the inclusion hyperspaces

$$\begin{aligned}S_\alpha(\Psi_{\mathcal{K}})^n(\mathcal{C})=\{&\mathcal{F}\underset{\text{cl}}{\subset} MX\mid\mathcal{F}\supset\\ &\supset\operatorname{Cl}(\bigcup_{(H,\mathcal{R}_1,\dots,\mathcal{R}_n)\in\mathcal{H}} M\mathcal{R}_1\circ M\mathcal{R}_2\circ\dots\circ M\mathcal{R}_n(F_{\mathcal{R}_1\dots\mathcal{R}_n}))\\ &\text{for an } n\text{-level } S_\alpha\mathcal{K}\text{-tree } \mathcal{H} \text{ and } F_{\mathcal{R}_1\dots\mathcal{R}_n}\in S_\alpha\mathcal{C}\}\end{aligned}$$

and

$$\begin{aligned}S_\alpha(\Psi_{\mathcal{K}})^n(\mathcal{C}')=\{&\mathcal{F}\underset{\text{cl}}{\subset} MX\mid\mathcal{F}\supset\\ &\supset\operatorname{Cl}(\bigcup_{(H,\mathcal{R}_1,\dots,\mathcal{R}_n)\in\mathcal{H}} M\mathcal{R}_1\circ M\mathcal{R}_2\circ\dots\circ M\mathcal{R}_n(F_{\mathcal{R}_1\dots\mathcal{R}_n}))\\ &\text{for an } n\text{-level } S_\alpha\mathcal{K}\text{-tree } \mathcal{H} \text{ and } F_{\mathcal{R}_1\dots\mathcal{R}_n}\in S_\alpha\mathcal{C}'\}\end{aligned}$$

is not greater than $q^n \operatorname{diam} X$. This implies that $\hat{\hat{d}}((\Psi_{\mathcal{K}})^n(\mathcal{C}), (\Psi_{\mathcal{K}})^n(\mathcal{C}')) \leqslant q^n \operatorname{diam} X$. □

Summing up, we obtain that the following theorem is true :

Theorem 5 (Fixed point theorem for distributions of capacities)**.** *Let (X, d) be a metric compactum and $\mathcal{K} \in M^2 R_q(X)$. Then $\Psi_{\mathcal{K}}$ is non-expanding, but is not a contraction. Nevertheless, there is a unique $\mathcal{C}_0$ such that $\Psi_{\mathcal{K}}(\mathcal{C}_0) = \mathcal{C}_0$, and for any $\mathcal{C} \in M^2 X$ we have $\hat{\hat{d}}((\Psi_{\mathcal{K}})^n(\mathcal{C}), \mathcal{C}_0) \leqslant q^n \operatorname{diam} X$.*

Thus we call $\mathcal{K}$ an *IFS* for distributions of capacities and $\mathcal{C}_0$ is an *attractor* of $\mathcal{K}$ or a *distribution of capacities that is self-similar w.r.t.* $\Psi_{\mathcal{K}}$.

Also *mutatis mutandis* :

Theorem 6 (Continuity of fixed points with respect to IFS for distributions of capacities)**.** *Let (X, d) be a metric compactum, and let $\mathcal{C}_0$, $\mathcal{C}_0'$ be attractors for $\mathcal{K}, \mathcal{K}' \in M^2 R_q(X)$ respectively. Then $\hat{\hat{d}}(\mathcal{C}_0, \mathcal{C}_0') \leqslant \sum_{n=1}^{\infty} \min\{\hat{\hat{d}}_u(\mathcal{K}, \mathcal{K}'), 2q^{n-1} \operatorname{diam} X\}$, therefore $\hat{\hat{d}}(\mathcal{C}_0, \mathcal{C}_0') \to 0$ as $\hat{\hat{d}}_u(\mathcal{K}, \mathcal{K}') \to 0$.*

Theorem 7 ("Collage+Anti-Collage theorem"for distributions of capacities)**.** *Let (X, d) be a metric compactum, $\mathcal{C} \in M^2 X$ and $\mathcal{K} \in M^2 R_q(X)$. If $\mathcal{C}_0$ is a fixed point of $\Psi_{\mathcal{K}}$, then*

$$\frac{1}{2}\hat{\hat{d}}(\mathcal{C}, \Psi_{\mathcal{K}}(\mathcal{C})) \leqslant \hat{\hat{d}}(\mathcal{C}, \mathcal{C}_0) \leqslant \sum_{n=1}^{\infty} \min\{\hat{\hat{d}}(\mathcal{C}, \Psi_{\mathcal{K}}(\mathcal{C})), q^{n-1} \operatorname{diam} X\}.$$

3 Final remarks

It is not difficult to describe a special case of IFS for distributions of capacities when the “big coefficient” is a $\cup$-capacity (= fuzzy set) of $\cup$-capacities (fuzzy sets). This case has a natural interpretation in terms of random grayscale images.

It is also straightforward to extend the presented results to fractal capacities with values in compact Lawson lattices (see [10]). For example, a color image in RGB mode can be regarded as an $\cup$-capacity with values in the lattice $[0; 1]^3$, so we expect that these results will be of practical importance. It is the topic of the next publication.

[1] *Choquet G.* Theory of Capacity // Ann. l'Institute Fourier. — 1953-1954. — **5**. — 131–295.

[2] *O'Brien G.L., Verwaat W.* How subsadditive are subadditive capacities? // Comment. Math. Univ. Carolinae. — 1994. — **35**(2). — 311–324.

[3] *Eichberger J., Kelsey D.* Non-additive beliefs and strategic equilibria // Games and Economic Behavior. — 2000. — **30**. — 183–215.

[4] *Epstein L., Wang T.* "Beliefs about beliefs" without probabilities // Econometrica. — 1996. — **64**(5). — 1343–1373.

[5] *Gilboa I., Schmeidler D.* Updating Ambiguous Beliefs // Journal of Economic Theory. — 1993. — **59**. — 33–49.

[6] *Schmeidler D.* Subjective Probability and Expected Utility without Additivity // Econometrica. — 1989. — **57**. — 571-—587.

[7] *Lin Zhou,* Integral representation of continuous comonotonically additive functionals // Trans. Amer. Math. Soc. — 1998. — **350**(5). — 1811–1822.

[8] *Zarichnyi M.M., Nykyforchyn O.R.* Capacity functor in the category of compacta // Mat. Sb. — 2008. — **199**:2. — 3–26.

[9] *Zarichnyi M., Teleiko A.* Categorical Topology of Compact Hausdorff Spaces. — Lviv, VNTL Publ., 1999.

[10] *Nykyforchyn O.R.* Capacities with values in compact Hausdorff lattices // Applied Categorical Structures. — 2008. — **15**(3). — 243–257.

[11] *Hlushak I.D., Nykyforchyn O.R.* Submonads of the capacity monad // Carpathian Journal of Mathematics. — 2008. — **24**:1. — 56–67.

[12] *Banach S.* Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrales // Fund. Math. — 1922. — **3**. — 133–181.

[13] *Centore P., Vrscay E.R.* Continuity properties for attractors and invariant measures for iterated function systems // Canadian Math. Bull. — 1994. — **37**. — 315–329.

[14] *Barnsley M.F., Ervin V., Hardin D., Lancaster J.* Solution of an inverse problem for fractals and other sets // Proc. Nat. Acad. Sci. USA. — 1985. — **83**. — 1975–1977.

[15] *Vrscay E.R., Saupe D.* "Can one break the 'collage barrier' in fractal image coding?" in *Fractals: Theory and Applications in Engineering*, ed. M. Dekking, J. Levy-Vehel, E. Lutton, and C. Tricot. — London, Springer Verlag, 1999, pp. 307–323.

[16] *Vrscay E.R.*, From Fractal Image Compression to Fractal-Based Methods in Mathematics, in *Fractals in Multimedia*, ed. by M. F. Barnsley, D. Saupe and E. R. Vrscay. — New York, Springer-Verlag, 2002.

[17] *Nykyforchyn O.R.* Geometry of spaces of capacities on metrizable compacta and components of multiplication of the capacity monad and two its submonads // Subm. to Math. Studii. — 2008. — 19 pp.

[18] *Forte B., Lo Schiavo M., Vrscay E.R.* Continuiuty properties of attractors for iterated fuzzy set systems // J. Austral. Math. Soc. Ser. B. — 1994. — **36**. — 175–193.

[19] *Hutchinson J.E., Rüshendorf L.* Random fractal measures via the contraction method // Indiana Univ. Math. J. — 1998. — **47**. — 471–487.

[20] *Forte B., Vrscay E.R.* Theory of generalized fractal transforms // NATO ASI on Fractal Image Encoding and Analysis, 8–17 July, 1995, Trondheim, Norway.

[21] *Cabrelli C.A., Forte B., Molter U.M., Vrscay E.R.* Iterated Fuzzy Set Systems: a new approach to the inverse problems for fractals and other sets // J. Math. Anal. and Appl. — 1992. — 79–100.

[22] *Barnsley M., Hutchinson J., Stenflo Ö.* A fractal valued random iteration algorithm and fractal hierarchy // Fractals. — 2005. — **13**(2). — 111–146.

[23] *Barnsley M., Hutchinson J., Stenflo Ö.* V-variable fractals and superfractals, preprint, 2003, `arXiv.org:math/0312314`.

ФРАКТАЛЬНІ ЄМНОСТІ ТА ІТЕРОВАНІ СИСТЕМИ ФУНКЦІЙ

Олег НИКИФОРЧИН

Прикарпатський національний університет
імені Василя Стефаника

Означено ітеровані системи функцій для гіперпросторів включення і ємностей, і доведено аналоги класичних теорем про атрактори, а саме теорему про нерухому точку, неперервність атрактора стосовно стискаючого відображення, а також Collage+Anti-Collage Theorem. Означено самоподібні випадкові ємності і вивчено їх властивості, аналогічні до властивостей випадкових самоподібних мір.