ON SUBGROUPS OF INFINITE INDEX

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Received September 23, 2008

The authors show that the set of all infinite (respectively infinite abelian) subgroups of infinite index of a group is either infinite or emtpy, and also describe groups with the minimal condition for abelian subgroups of infinite index.

All notations throughout the paper are standard.

The results of the present paper are the following four theorems.

Theorem 1. *Let G be a group. Then the following statements hold.*

(i) *If G has some infinite subgroup K of infinite index then the set of all infinite subgroups of infinite index of G is infinite.*

(ii) *If G has some infinite abelian subgroup A of infinite index then the set of all infinite abelian subgroups of infinite index of G is infinite.*

Theorem 2. *Let G be a nonperiodic group. Then the following statements are equivalent.*

(i) *G satisfies the minimal condition for subgroups of infinite index.*

(ii) *G satisfies the minimal condition for abelian subgroups of infinite index.*

(iii) *$|G : H| < \infty$ for any infinite cyclic subgroup H of G.*

(iv) *G is almost infinite cyclic.*

(v) *G is finite-by-infinite cyclic or finite-by-infinite dihedral.*

(vi) *G has no infinite subgroups of infinite index.*

(vii) *G has no infinite abelian subgroups of infinite index.*

Theorem 3. *Let G be a group. Then the following statements hold.*

(i) *G satisfies the minimal condition for abelian subgroups of infinite index if and only if either it is almost infinite cyclic or all its abelian subgroups are Chernikov.*

(ii) *G has no infinite abelian subgroups of infinite index if and only if either it is almost (quasicyclic or infinite cyclic), or all its abelian subgroups are finite.*

Recall that a group G is called *Shunkov* (or *conjugatively fiprimitively finite*) if $\langle g, g^h \rangle$ is finite for every its finite subgroup K and elements $g, h \in N_G(K)$, whenever g is of prime order. The class of all Shunkov groups is wide and contains, for instance, all locally finite, binary finite, 2-groups and all torsion-free groups. A great many deep results connected with Shunkov groups are collected in [1].

Theorem 4. *Let G be a locally solvable or* 2- *or Shunkov group. Then the following statements hold.*

(i) *G satisfies the minimal condition for abelian subgroups of infinite index if and only if it is either Chernikov or infinite cyclic-by-finite (respectively it is Chernikov, it is either Chernikov or finite-by-infinite cyclic).*

(ii) *G has no infinite abelian subgroups of infinite index if and only if it is almost quasicyclic or infinite cyclic-by-finite, or finite (respectively it is almost quasicyclic or finite, it is almost quasicyclic or finite-by-infinite cyclic, or finite).*

(iii) *G has no infinite abelian subgroups of infinite index if and only if it has no infinite subgroups of infinite index.*

Remark. A. Yu. Ol'shanskiĭ's examples of infinite nonabelian groups with finite proper subgroups (see, for instance, [2]) demonstrate that for an arbitrary group G each statement of Theorem 4 is false.

Proof of Theorem 1. (i) If $|G : N_G(K)|$ is infinite then the set of all distinct subgroups K^g, $g \in G$, is infinite.

Let $|G : N_G(K)|$ be finite. Then $N_G(K)/K$ is infinite. Therefore, the set of all cyclic subgroups C/K of $N_G(K)$ is infinite. An arbitrary C is clearly infinite. If G is periodic then $|G : C|$ is infinite. Thus, in this case the set of all infinite subgroups of infinite index of G is infinite.

Suppose that $N_G(K)/K$ contains some infinite cyclic subgroup B/K. Then $B = KD$ and $K \cap D = 1$ for some infinite cyclic subgroup D of B. Therefore, $|G : D|$ is infinite and the set of infinite subgroups of D is infinite. Thus, in this case the set of all infinite subgroups of infinite index of G is infinite.

(ii) Obviously, we may assume that the set of infinite subgroups of A is finite. So, A satisfies the minimal condition for subgroups. Consequently, A is Chernikov. So, A contains some subgroup B of finite index which is a direct product of quasicyclic subgroups. Since the set of all infinite subgroups of B is finite, B is obviously quasicyclic. Further, in the case when $|G : N_G(B)|$ is infinite, the set of all distinct subgroups B^g, $g \in G$ is infinite. At the same time, the set of all infinite abelian subgroups of G of infinite index is infinite.

Let $|G : N_G(B)| < \infty$. First, suppose that G has some infinite cyclic subgroup H. Since B is infinite periodic, $H \cap B = 1$ and simultaneously $|G : H|$ is infinite. Therefore, since the set of infinite subgroups of H is infinite, the set of infinite abelian subgroups of G is infinite.

Suppose that G is periodic. Then $|N_G(B) : C_G(B)| < \infty$, because B is quasicyclic. So, $|G : C_G(B)| < \infty$. Since $|G : B|$ is infinite, $C_G(B)/B$ is infinite as well. Therefore, the set of all cyclic subgroups C/B of $C_G(B)/B$ is infinite. Then the set of all subgroups C is infinite. Clearly, any C is infinite abelian and has an infinite index in G.

The theorem is proven.

Proof of Theorem 2. Clearly, (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii) Since H does not satisfy the minimal condition for abelian subgroups and G satisfies the minimal condition for abelian subgroups of infinite index, we obtain that $|G : H| < \infty$.

(iii) $\Rightarrow$ (iv). In view of Poincare's theorem, H contains some normal subgroup of G of finite index. This subgroup is infinite cyclic.

(iv) $\Rightarrow$ (v). Let T be a maximal normal periodic subgroup of G. Since G is almost infinite cyclic, T is finite. The quotient group G/T clearly is almost infinite cyclic and has no nonidentity normal periodic subgroups. Taking this into account, we may assume without loss of generality that $T = 1$. Then G has no nonidentity normal periodic subgroups.

Let N be a normal infinite cyclic subgroup of finite index of G. Then

$$|G : C_G(N)| \leq 2. \qquad (1)$$

Let g be an element of $C_G(N)$ of finite order. Since $C_G(g) \supseteq N$, one has that $|G : C_G(g)| < \infty$. Therefore, the set $R = \{g^x \colon x \in G\}$ is finite. Since R consists of finite order elements of G and $R^y = R$, $y \in G$, we have that $\langle R \rangle$ is a finite normal subgroup of G (Dietzmann's Lemma [3], see also [4]). Consequently, $\langle R \rangle = 1$. So, $g = 1$. Thus, $C_G(N)$ is torsion-free.

Let F be either a subgroup of order > 2 of G, or a nonidentity subgroup of $C_G(N)$. Then with regard to (1), we obtain that $F \cap C_G(N) \neq 1$. Since $C_G(N)$ is torsion-free, $F \cap C_G(N)$ is infinite. Therefore $|G : N| < \infty$ yields that $F \cap C_G(N) \cap N = F \cap N$ is infinite. So, $F \cap N \neq 1$. Since N is cyclic, we have that $|N : F \cap N| < \infty$. On the other hand, $\infty > |G : F \cap N| \geq |G : F| \geq |C_G(N) : F \cap C_G(N)|$. In view of Fedorov's theorem [5] (see also Theorem 4.33 of [4]), $C_G(N)$ is infinite cyclic and also in the case when G is not infinite cyclic, it contains some subgroup T of order 2. Since $C_G(N)$ is infinite cyclic and (1) holds, in the last case, $G = C_G(N) \leftthreetimes T$. Then because T is not normal in G, the group G is infinite dihedral.

(v) $\Rightarrow$ (vi). Let L be a finite normal subgroup of G for which G/L is infinite cyclic or infinite dihedral, and K be an arbitrary subgroup of G of infinite index. Then KL/L is a subgroup of G/L of infinite index. Moreover, G/L has an infinite cyclic subgroup H of finite index. Then $|H : H \cap KL/L|$ is infinite. Consequently, $H \cap KL/L = L$ and so, KL/L is finite. Hence, it follows that K is finite.

Clearly, (vi) $\Rightarrow$ (vii).

(viii) $\Rightarrow$ (i). Indeed, G contains an infinite cyclic subgroup of finite index. Therefore G, as G/L above, has no infinite subgroups of infinite index. At the same time, (i) is valid.

Theorem is proven.

Proof of Theorem 3. (i) N e c e s s i t y . If G is not periodic, then in view of Theorem 2, it is almost infinite cyclic.

Suppose that G is periodic. Let A be some of its abelian subgroup. If A is finite then it is Chernikov. Suppose that A is infinite. Put $B = \langle a \colon a \in A$ and $|\langle a \rangle|$ is prime$\rangle$. Let a_1 be an element of B of prime order; if $B \neq \langle a_1 \rangle$ then a_2 is an element of prime order from $B \setminus \langle a_1 \rangle$; if $B \neq \langle a_1 \rangle \langle a_2 \rangle$ then a_3 is an element of prime order from $B \setminus \langle a_1 \rangle \langle a_2 \rangle$; .... In the case when B is finite, for some n, $B = \mathop{\times}\limits_{i=1}^{n} \langle a_i \rangle$. In the case when B is infinite, $B_1 = \langle a_1, a_3, a_5, a_7, \ldots \rangle$ is an infinite subgroup of infinite index of B. Analogously, B_1 contains some infinite subgroup B_2 of infinite index, B_2 contains some infinite subgroup B_3 of infinite index and so on, which is a contradiction.

Thus, $B = \mathop{\times}\limits_{i=1}^{n} \langle a_i \rangle$.

Let A_1 be maximal among all subgroups X of $A = A_0$ such that $X \cap \langle a_1 \rangle = 1$ and also $a_2, \dots, a_n \in X$ if $n > 1$. Obviously, $\langle a_1 \rangle A_1/A_1$ is the unique subgroup of A_0/A_1 of prime order. Then clearly for some prime p, A/A_1 is a p-group. If A_0/A_1 is neither cyclic nor quasicyclic then, obviously, there exist elements g, $h \in A_0/A_1$ such that $g^p = h^p$ and $\langle g \rangle \neq \langle h \rangle$. Then $\langle gh^{-1} \rangle$ is of order p and $\langle gh^{-1} \rangle \cap \langle a_1 \rangle A_1/A_1 = 1$, which is a contradiction. Thus, A_0/A_1 is cyclic or quasicyclic. Analogously, there exists a similar subgroup A_2 of A_1, if $n > 1$, A_3 of A_2, if $n > 2$, ... Clearly, $A_n = 1$.

Let $l = 1$, if all quotient groups $A_0/A_1, \dots, A_{n-1}/A_n$ are qasicyclic, and otherwise l be the product of orders of all cyclic groups among $A_0/A_1, \dots, A_{n-1}/A_n$.

Let φ be any homomorphism of G, for which G^φ has a finite exponent. Then for any $0 \le i < n$, $|A_i^\varphi/A_{i+1}^\varphi| \le |A_i/A_{i+1}|$, and also $|A_i^\varphi/A_{i+1}^\varphi| = 1$, if A_i/A_{i+1} is quasicyclic. Consequently, $|G^\varphi| \le l$. Since for any subgroups X and Y of G with G/X and G/Y of finite exponent, $G/X \cap Y$ has a finite exponent, for the intersection H of all such X, one has that $|G/H| \le l$. Let q be any prime and $Q = \{h^q\colon h \in H\}$. Since H/Q is of finite exponent, G/Q is of finite exponent as well. Consequently, $H = Q$.

Thus, H is divisible. Therefore, as it is well known, H is a direct product of quasicyclic subgroups. Since H has finitely many subgroups of prime order, the number of direct multipliers is finite. Thus, an arbitrary abelian subgroup A of G is Chernikov.

S u f f i c i e n c y . If all abelian subgroups of G are Chernikov then, of course, it satisfies the minimal condition for abelian subgroups of infinite index. If G is almost infinite cyclic then in view of Theorem 2, it satisfies the last condition too.

(ii) N e c e s s i t y . Suppose that G is not almost cyclic and contains some infinite abelian subgroup A. In view of (i), A contains a subgroup B of finite index which is a direct product of quasicyclic subgroups. Since B has no infinite subgroups of infinite index, it is quasicyclic. Further, $|G : B| < \infty$ and B has no proper subgroups of finite index. Consequently, in view of Poincare's theorem, B is normal in G. Thus, G is almost quasicyclic.

S u f f i c i e n c y . Suppose that G contains a subgroup A of finite index, which is quasicyclic or infinite cyclic. Let H be a subgroup of infinite index of G. Then $|A : H \cap A|$ is infinite. Therefore, $H \cap A$ is finite. Since $|G : A| < \infty$, H is finite.

The theorem is proven.

Proof of Theorem 4. (i) Necessity. First, by Theorem 3 (i), either G is infinite cyclic-by-finite or all its abelian subgroups are Chernikov. Further, in view of Theorem 2, an infinite cyclic-by-finite group is finite-by-infinite (cyclic or dihedral). Since any two distinct elements of order 2 of infinite dihedral group generate an infinite subgroup, a finite-by-infinite dihedral group is not Shunkov. Thus, a Shunkov infinite cyclic-by-finite group is finite-by-infinite cyclic.

Finally, respectively in view of the theorems of S. N. Chernikov [6] (see also Theorem 4.3 of [7]), Shunkov [8], Suchkova – Shunkov [9] (see also Theorem 4.5.1 of [1]), locally solvable, 2-, Shunkov groups with Chernikov abelian subgroups are Chernikov.

Sufficiency. If G is Chernikov then it satisfies the minimal condition for abelian subgroups of infinite index, of course. If G is infinite cyclic-by-finite or finite-by-infinite cyclic then, in view of Theorem 2, it satisfies this condition.

(ii) Necessity is an easy consequence of (i). Sufficiency is a consequence of Theorem 3 (ii) and Theorem 2 (the equivalence (iv) $\Leftrightarrow$ (v)).

(iii) Necessity. In view of (ii), G is one of the following: finite, almost quasicyclic, infinite cyclic-by-finite, finite-by-infinite cyclic. Any infinite subgroup of a group of the second type contains obviously a quasicyclic subgroup and, at the same time, has a finite index. In view of Theorem 2, groups of third and fours types also have no infinite subgroups of infinite index.

Sufficiency is obvious.

Theorem is proven.

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ПРО ПІДГРУПИ НЕСКІНЧЕННОГО ІНДЕКСУ

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