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CONDITIONS FOR CONSTRUCTING A SQUARE MATRIX THAT CONTAINS A SQUARE SUBMATRIX WITH GIVEN INVARIANT FACTORS

Necessary and sufficient conditions that $(n-k)\times(n-k)$*-matrix* A, $1 \le k \le n-1$, *may be augmented to* $n \times n$*-matrix* C *with given invariant factors over elementary divisor domains are established. Moreover, we indicated some properties of the invariant factors of matrix* C *and its submatrix* A.

Key words: elementary divisor domain, square matrix, invariant factors, Smith normal form.

An important role in modern algebra plays the problem in the studying of the arithmetic properties of matrices. Invariant factors, their properties, relationships and divisibility are widely used for these studies [1, 5, 6, 7, 9, 11]. In particular, at augmented one matrix with a single row to obtain another matrix are used the relationships between the invariant factors of these matrices. Based on the relationships between the invariant factors of matrix and its submatrix B. Jones [3], asserted that a unimodular $m \times n$ $(m < n)$ matrix A over a principal ideal domain may always be augmented with a single row to obtain a unimodular $(m+1)\times n$ matrix C.

Over the same area, for arbitrary matrix, R. Thompson [10] showed some relationships between the invariant factors of a matrix A and those of a one row prolongation C. Note that the factoriality properties of the ring were used to prove these results. Similar results in terms of a finitely generated module was obtained by D. Carson [2] over principal ideal domains.

V. Shchedryk [8, Proposition 3.7] indicated necessary and sufficient conditions that an $m \times s$ matrix A with given invariant factors is a submatrix of some an invertible $n \times n$ matrix over elementary divisor domain. Over the same area, A. Romaniv and N. Dzhaliuk [6] extended Thompson's results to elementary divisor domain. In particular, they obtained necessary and sufficient conditions that a matrix A may be augmented with a single row to obtain a matrix C.

Recall, that R be an elementary divisor domain [4], if every $m \times n$ matrix A over R has diagonal reduction. Namely

$$A \sim \mathrm{E} = \operatorname{diag}(\varepsilon_1,\ldots,\varepsilon_s,0,\ldots,0), \quad \varepsilon_i \,|\, \varepsilon_{i+1}, \quad i = 1,\ldots,s-1,$$

where the matrix E is called the Smith normal form, ε_i are invariant factors of the matrix A. We have the following result.

Theorem 1. (Theorem 3 in [6]) Let R be an elementary divisor domain, A be an $(n-1)\times(n-1)$ matrix over R, $A \sim \mathrm{E} = \operatorname{diag}(\varepsilon_1,\ldots,\varepsilon_s,0,\ldots,0)$, $\varepsilon_i \,|\, \varepsilon_{i+1}$, $i = 1,\ldots,s-1$, $\operatorname{rang} A = s \le n-1$. Then an $n \times n$ matrix $C \sim \Gamma = \operatorname{diag}(\gamma_1,\ldots,\gamma_t,0,\ldots,0)$, $\gamma_i \,|\, \gamma_{i+1}$, $i = 1,\ldots,t-1$, $\operatorname{rang} C = t \le n$, exists and contains the matrix A as a submatrix if and only if

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$$\begin{aligned} &\gamma_1 | \varepsilon_1 | \gamma_3 , \\ &\gamma_2 | \varepsilon_2 | \gamma_4 , \\ &\dots\dots\dots\dots \\ &\gamma_{s-1} | \varepsilon_{s-1} | \gamma_{s+1} , \\ &\gamma_s | \varepsilon_s . \end{aligned} \tag{1}$$

Naturally, the problem arises, is this result true for matrix of the order $(n-k)\times(n-k)$, where $1\le k\le n-1$? This question was asked by Thompson [10], namely: When can C be constructed such that A is a submatrix? Based the factoriality properties of the ring this problem for a square matrces was solved in [10] over principal ideal domains.

In this paper these results was extended to elementary divisor domain. Namely, some properties of the invariant factors of matrix and its submatrix are established. Based on obtained properties, we give necessary and sufficient conditions for the existence of an $n\times n$ –square matrix C containing $(n-k)\times(n-k)$, $1\le k\le n-1$ –square matrix A as a submatrix such that both C and A have given invariant factors.

Let R be an elementary divisor domain [4] with $1\ne 0$. And let C be an $n\times n$ matrix over R. The matrix C has diagonal reduction

$$C \sim \Gamma = \mathrm{diag}(\gamma_1,\dots,\gamma_t,0,\dots,0),\ \gamma_i | \gamma_{i+1},\ i=1,\dots,t-1.$$

Let A be an $(n-k)\times(n-k)$ matrix over R and has diagonal reduction

$$A \sim \mathrm{E} = \mathrm{diag}(\varepsilon_1,\dots,\varepsilon_s,0,\dots,0),\ \varepsilon_i | \varepsilon_{i+1},\ i=1,\dots,s-1.$$

The notation $a|b$ means that the element a divides the element b, and by the symbol $[a,b]$ we denote the least common multiple of the elements a and b.

Theorem 2. Let R be an elementary divisor domain, A be an $(n-k)\times(n-k)$, $1\le k\le n-1$ matrix over R, $A\sim \mathrm{E}=\mathrm{diag}(\varepsilon_1,\dots,\varepsilon_s,0,\dots,0)$, $\varepsilon_i | \varepsilon_{i+1}$, $i=1,\dots,s-1$, $\mathrm{rang}A = s\le n-k$. Then an $n\times n$ matrix $C\sim\Gamma=\mathrm{diag}(\gamma_1,\dots,\gamma_t,0,\dots,0)$, $\gamma_i|\gamma_{i+1}$, $i=1,\dots,t-1$, $\mathrm{rang}C = t\le n$ exists and contains the matrix A as a submatrix if and only if

$$\begin{aligned} &\gamma_1 | \varepsilon_1 | \gamma_{1+2k} , \\ &\gamma_2 | \varepsilon_2 | \gamma_{2+2k} , \\ &\dots\dots\dots\dots \\ &\gamma_s | \varepsilon_s | \gamma_{s+2k} , \end{aligned} \tag{2}$$

and

$$\gamma_{s+k+1} = \gamma_{s+k+2} = \dots = \gamma_{s+2k} = 0. \tag{3}$$

Proof. Necessity. Let the matrix C exists and contains as a submatrix the matrix A.

Since A has the order $(n-k)\times(n-k)$, where $1\le k\le n-1$, then we have a nested chain of submatrices of C

$$A_{n-k}\subset A_{n-k+1}\subset\dots\subset A_n.$$

Denote by B_{n-k+j}, $j=0,\dots,k$ the matrices $A_{n-k},\dots,A_n$. Obvious, that $B_{n-k}=A_{n-k}=A$ and $B_n=A_n=C$. Then B_{n-k+j} has the invariant factors

$\delta_1,\ldots,\delta_{s+j}$, where $\delta_1 | \delta_2 | \ldots | \delta_{s+j}$, $j=0,\ldots,k$. Obvious, that

$$\delta_{s+j+1} = \delta_{s+j+2} = \ldots = 0, \; j = 0,\ldots,k.$$

Note, that for $j=0$ and $j=1$ we get, that $B_{n-k} = A$ has the invariant factors $\varepsilon_1,\ldots,\varepsilon_s$ and B_{n-k+1} has the invariant factors $\delta_1,\ldots,\delta_s,\delta_{s+1}$. Then according to Theorem 1 we have the condition

$$\delta_i | \varepsilon_i | \delta_{i+2}, \; i = 1,\ldots,s.$$

Similarly, consider the invariant factors $\delta_1,\ldots,\delta_{s+j}$ of B_{n-k+j} and the invariant factors $\delta_1^*,\ldots,\delta_{s+j}^*,\delta_{s+j+1}^*$ of $B_{n-k+j+1}$, $j=0,\ldots,k$. According to Theorem 1 we have:

$$\delta_i^* | \delta_i | \delta_{i+2}^*, \; i = 1,\ldots,s+j.$$

From this it easily follows that

$$\gamma_1 | \varepsilon_1 | \gamma_{1+2k},$$
$$\gamma_2 | \varepsilon_2 | \gamma_{2+2k},$$
$$\ldots\ldots\ldots\ldots$$
$$\gamma_s | \varepsilon_s | \gamma_{s+2k},$$

where

$$\gamma_{s+k+1} = \gamma_{s+k+2} = \cdots = \gamma_{s+2k} = 0.$$

S u f f i c i e n t. For proof we use the method of induction. Assume $k=1$, then the matrix A has the order $(n-1)\times(n-1)$ and we have the case of Theorem 1.

Let $k>1$. Then suppose that the matrix A with the Smith normal form E and the Smith normal form Γ are given such that (2) and (3) hold. Consider such nonzero elements $\delta_1,\cdots,\delta_{s+1} \in R$, that

$$\delta_i | \varepsilon_i | \delta_{i+2}, \; i = 1,\ldots,s, \tag{4}$$

$$\gamma_i | \delta_i | \gamma_{i+2(k-1)}, \; i = 1,\ldots,s+1. \tag{5}$$

Note, that $\delta_{s+2} = 0$.

Consider $(n-k+1)\times(n-k+1)$ matrix B, where

$$B \sim \Delta = \mathrm{diag}(\delta_1,\ldots,\delta_{s+1},0\ldots,0), \; \delta_i | \delta_{i+1}, \; i = 1,\ldots,s,$$

where δ_i – invariant factors of B. If the condition (4) hold, then according to Theorem 1 the matrix B contains as a submatrix the matrix A. If the condition (5) hold, by the case $k-1$ of Theorem 2, then the matrix C contains as a submatrix the matrix B. Therefore, for proof Theorem it is necessary to select such nonzero elements $\delta_1 | \delta_2 | \;,\cdots, | \delta_{s+1}$ that conditions (4) and (5) hold.

Choose such the elements $\delta_i \in R$, $i = 1,\ldots,s+1$ that

$$\delta_i = [\gamma_i, \varepsilon_{i-2}], \; i = 1,\ldots,s+1, \tag{6}$$

where it is understood that $\varepsilon_{-1} = \varepsilon_0 = 1$.

Obvious, that from (6) we get the following divisibility $\delta_1 | \delta_2 | \cdots | \delta_{s+1}$, $\gamma_i | \delta_i$ for $i = 1, \ldots, s+1$ and $\varepsilon_{i-2} | \delta_i$ for $i = 1, \ldots, s$. From (2) we get, that $\varepsilon_i | \gamma_{i+2k}$ and $\gamma_i | \varepsilon_i$. And this means, taking into account Theorem 1 that

$$\delta_i | \gamma_{i+2(k-1)}, \quad i = 1, \ldots, s+1,$$

and

$$\delta_i | \varepsilon_i, \quad i = 1, \ldots, s.$$

Therefore $\delta_1, \cdots, \delta_{s+1}$ satisfy (4) and (5). This completes the proof of Theorem 2. ◊

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УМОВИ ПОБУДОВИ КВАДРАТНОЇ МАТРИЦІ, ЯКА МІСТИТЬ КВАДРАТНУ ПІДМАТРИЦЮ ІЗ ЗАДАНИМИ ІНВАРІАНТНИМИ МНОЖНИКАМИ

Встановлено необхідні та достатні умови доповнення $(n-k)\times(n-k)$*-матриці* A, $1 \le k \le n-1$ *до* $n \times n$*-матриці* C *із заданими інваріантними множниками над областями елементарних дільників. Більше того, вказано деякі властивості інваріантних множників матриці* C *та її підматриці* A.

Ключові слова: *область елементарних дільників, квадратна матриця, інваріантні множники, нормальна форма Сміта.*

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