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ON HASSE DIAGRAMS CONNECTED WITH THE 1-OVERSUPERCRITICAL POSET (1, 2, 7)

We study combinatorial properties of Hasse diagrams of posets minimax equivalent to the oversupercritical poset (1, 2, 7).

Key words: *oriented graph, direct sum, 0-length, Hasse diagram, isomorphism, anti-isomorphism, 1-oversupercritical poset, minimax equivalence.*

Introduction. M. M. Kleiner proved that a (finite) poset S is of finite representation type if and only if it does not contain the subsets of the form (1, 1, 1, 1), (2, 2, 2), (1, 3, 3), (1, 2, 5) and (И, 4), which are called the critical posets (see [7]). By (P,Q) with P, Q being posets we denote their direct sum, and by $(i_1, i_2, \ldots, i_p)$ the direct sum of chains of length $i_1, i_2, \ldots, i_p$. In [4] it is proved that a poset is P-critical (i.e. critical with respect to the positivity of the Tits quadratic form) if and only if it is minimax equivalent to a critical poset (the notion of minimax equivalence was introduced in [1]); in [4] all such posets were completely described (up to dyality).

A similar situation occurs for the tame posets. L. A. Nazarova proved that a poset S is tame if and only if it does not contain subsets of the form (1, 1, 1, 1, 1), (1, 1, 1, 2), (2, 2, 3), (1, 3, 4), (1, 2, 6) and (И, 5) (see [8]); these sets are called supercritical. In [5] it is proved that a poset is critical with respect to the non-negativity of the Tits form if and only if it is minimax equivalent to a supercritical poset; all such critical posets were described in [6].

The posets that are differ from the supercritical sets in the way as the supercritical sets differ from critical posets are called 1-supersupcritical. More precisely (see [2]), it is the following posets: 1) (1, 1, 1, 1, 1, 1), 2) (1, 1, 1, 1, 2), 3) (1, 1, 2, 2), 4) (1, 1, 1, 3), 5) (2, 3, 3), 6) (2, 2, 4), 7) (1, 4, 4), 8) (1, 2, 7), 9) (1, 2, 7), 10) (6, И).

We study combinatorial properties of the posets which are minimax equalently to the 1-oversupercritical poset (1, 2, 7). For the smallest 1-over-supercritical poset with trivial group of automorphisms, i. e. the poset (1, 3, 5), combinatorial properties were studied in [3].

1. Formulation of the main theorems. The Hasse diagram of a finite poset S is by definition the directed graph $H(S)$ with the vertices $x \in S$ and the arrows (x, y), $x, y \in S$, where $x < y$ and there is no z satisfying $x < z < y$. We call the 0-length of an oriented path of $H(S)$ the number of its vertices, and denote by $l_{\min}(S)$ (respectively, $l_{\max}(S)$) the 0-length of a most short (respectively, long) oriented path of $H(S)$. We denote by [S] – the set of all posets minimax equivalent to S (see [1]) and put

$$L_{\min}(S) = \min_{X \in [S]^{\sim}} l_{\min}(X), \qquad L_{\max}(S) = \max_{X \in [S]^{\sim}} l_{\max}(X).$$

The number of minimax oriented paths of the graph $H(X)$ (X is a poset) is denoted by $n(X)$.

Theorem 1. *For $S=(1,2,7)$, $L_{\min}(S) = 1$, $L_{\max}(S) = 9$.*

Theorem 2. *Let $X \in [(1,2,7)]$. Then* $1 \le l_{\min}(X) \le 8$, $5 \le l_{\max}(X) \le 9$, $n(X) \in \{2, 3, 4\}$.

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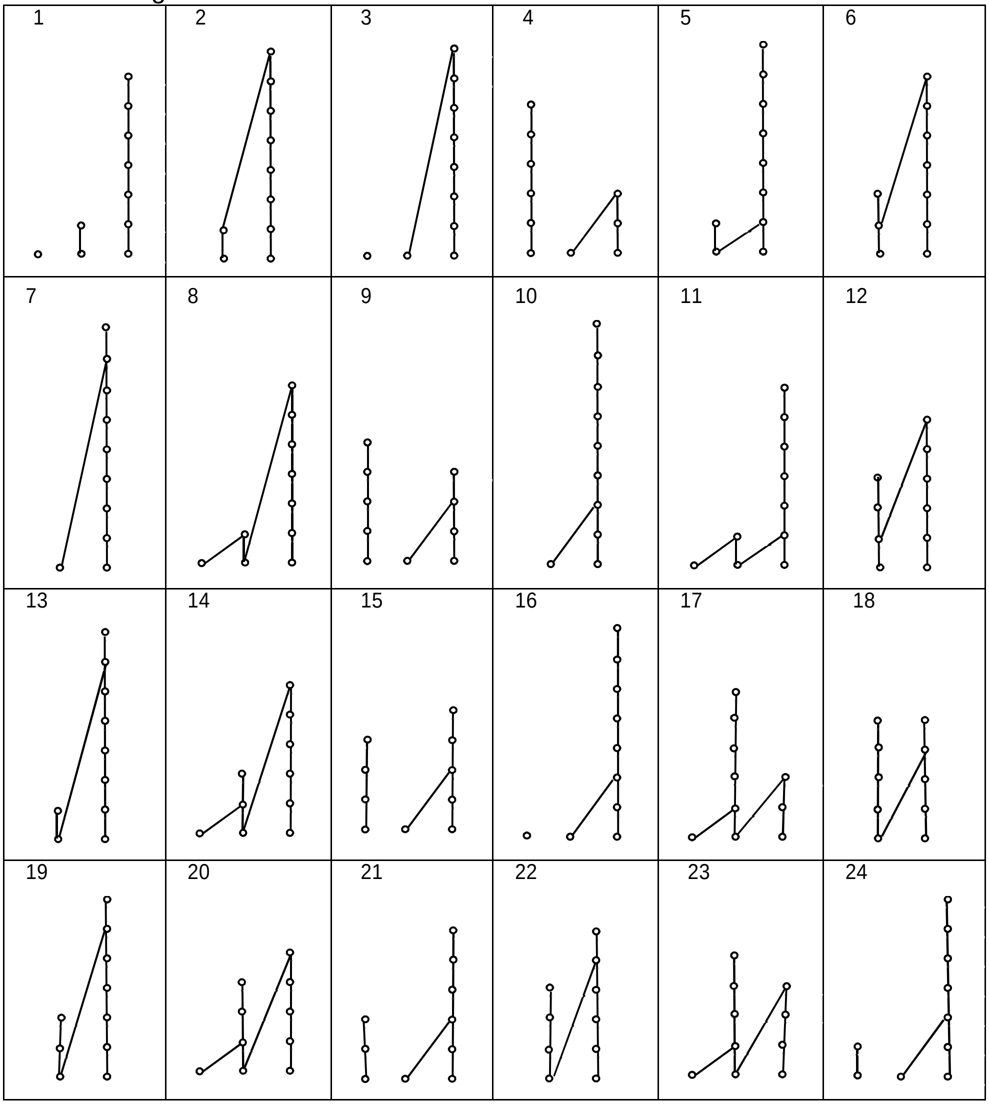
2. Proofs of the theorems. The posets minimax equivalent to (1, 2, 7) were classified in [2]. They are given (up to isomorphism and anti-isomorphism) by the following table.



We have the next theorem.

Theorem 3. *The following holds for posets* 1 – 24:

N	$l_{\min}$	$l_{\max}$	n	N	$l_{\min}$	$l_{\max}$	n	N	$l_{\min}$	$l_{\max}$	n
1	1	7	3	9	3	5	3	17	2	6	4
2	3	8	2	10	8	9	2	18	3	5	3
3	1	8	3	11	2	7	4	19	3	7	3
4	2	6	3	12	3	6	3	20	2	5	4
5	2	8	3	13	2	8	3	21	3	6	3
6	3	7	3	14	2	6	4	22	3	6	3
7	3	9	2	15	4	5	3	23	2	5	4
8	2	7	4	16	1	8	3	24	2	7	3

Here the numbers $I_{\min}$, $I_{\max}$ and n denote, respectively, $I_{\min}(S_i)$, $I_{\max}(S_i)$ and $n(S_i)$ for $i = N$, where N runs from 1 to 24 and S_i is the i th poset in the table.

The proof is carried out by direct calculations.

Theorems 1 and 2 follow from Theorem 3.

1. *Bondarenko V. M.* On (min, max)-equivalence of posets and applications to the Tits forms // Bull. of the University of Kiev (series: Physics & Mathematics). – 2005. – № 1. – P. 24–25.
2. *Bondarenko V. V., Bondarenko V. M., Stepochkina M. V., Chervyakov I. V.* 1-oversupercritical partially ordered sets with trivial group of automorphisms and min-equivalence. I. // Nauk. Visn. Uzhgorod. Univ., Ser. Mat. – 2011. – 22, № 2. – P. 17–25 (in Russian).
3. *Bondarenko V. M., Orlovskaja Yu. M., Styopochkina M. V.* On Hasse diagrams connected with the 1-oversupercritical poset (1, 3, 5) // Appl. problems of mech. and math. – 2018. – 16. – P. 30–32.
4. *Bondarenko V.M., Stepochkina M.V.* (Min, max)-equivalence of partially ordered sets and the Tits quadratic form // Zb. Pr. Inst. Mat. NAN Ukr. / Problems of Analysis and Algebra / – 2005. – 2, № 3. – P. 18–58 (in Russian).
5. *Bondarenko V. M., Stepochkina M. V.* (Min, max)-equivalency of posets and nonnegative Tits forms // Ukrainian Math. J. – 2008. – 60, № 9. – P. 1349–1359.
6. *Bondarenko V. M., Stepochkina M. V.* Description of posets critical with respect to the nonnegativity of the quadratic Tits form // Ukrainian Math. J. – 2009. – 61, № 5. – P. 734–746.
7. *Kleiner M. M.* Partially ordered sets of finite type // Zap. Nauch. Semin. LOMI. – 1972. – 28. – P. 32–41 (in Russian).
8. *Nazarova L. A.* Partially ordered sets of infinite type // Izv. Akad. Nauk SSSR Ser. Mat. – 1975. – 39, № 5. – P. 963–991 (in Russian).

ПРО ДІАГРАМИ ХАССЕ, ПОВ'ЯЗАНІ З 1-НАДСУПЕРКРИТИЧНОЮ ЧАСТКОВО ВПОРЯДКОВАНОЮ МНОЖИНОЮ (1, 2, 7)

Вивчено комбінаторні властивості діаграм Хассе частково впорядкованих множин, мінімаксно еквівалентних надсуперкритичній частково впорядкованій множині (1, 2, 7).

Ключові слова: *орієнтований граф, пряма сума, 0-довжина, діаграма Хассе, ізоморфізм, анти-ізоморфізм, 1-надсуперкритична ч. в. множина, мінімаксна еквівалентність.*

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