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CLASSIFICATION OF GROUP ISOTOPES ACCORDING TO THEIR INVERSE PROPERTIES

Coincidence of translation sets of the same directions in a quasigroup defines nine varieties: IP, CIP and mirror quasigroup varieties [9]. Their intersection with the variety of group isotopes is studied. In particular, it is proved that in the variety of group isotopes, the subvarieties of the middle, left and right mirror quasigroups coincide with the subvarieties of commutative, left and right symmetric quasigroups respectively.

Key words: quasigroup, identity, variety, parastrophe, group isotope.

Introduction. This article is a continuation of the works [8, 9]. In each quasigroup Q, six types of translations: the left, right and middle translations and their inverses are defined. Two translations may coincide as permutations of Q, and yet be different when considered upon the web of the quasigroup. In [8] each of the translation types will be called a direction. Properties of the directions are considered in [8]. Coincidence of translation sets of the same directions in a quasigroup defines nine quasigroup varieties. Four of them: LIP, RIP, MIP and CIP are well known. The remaining five quasigroup varieties are relatively new because they are left and right inverses of CIP variety and the generalization of commutative, left and right symmetric quasigroups.

The classes of quasigroups which are isotopic to groups was under consideration in the works [1, 2, 4–6, 8, 12, 13] and many others. The theory of group isotopes was systematized in the works "On group isotopes" [4–6] by F. Sokhatsky. The isotopic closure of some group varieties was studied by G. Belyavskaya [14], A. Drápal [15], A. Tabarov [16]. The structure of CI quasigroups for which all LP-isotopes are CI-loops was investigated in [2] by V. Belousov, B. Tsurkan.

According to the concept of parastrophic symmetry introduced by F. Sokhatsky [11], the class of all quasigroups is divided into six classes: the class of all asymmetric quasigroups and five varieties of quasigroups (commutative, left symmetric, right symmetric, semi-symmetric and totally symmetric). Each of these classes is characterized by symmetry groups of its quasigroups [7].

Here, the conditions under which these varieties of IP, CIP and mirror quasigroups are isotopic groups are found.

The parastrophy orbits of quasigroups with inverse properties are described in the works of F. Sokhatsky, A. Lutsenko [6, 8, 9]. In particular, isotopes of the groups that are left, right (in the case $\lambda 0 = 0$ and $\rho 0 = 0$ respectively), and middle IP quasigroups are described. The isotopes of the groups that are left and right IP quasigroups (in the case $\lambda 0 \neq 0$, $\rho 0 \neq 0$) are investigated. In [9], a parastrophy orbit of varieties with a cross inverse property and a parastrophy orbit of varieties of mirror quasigroups are found.

In [3], it is proved that a left linear quasigroup $(Q;\cdot)$ over a loop $(Q;+)$ is a CIP quasigroup with the inversion mapping $\gamma 0 = 0$. In this paper, we investigate a more general case.

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1. Preliminaries. An algebra $(Q;\ \circ;\ \overset{\ell}{\circ};\ \overset{r}{\circ})$ with identities

$$(x\circ y)\overset{\ell}{\circ}y=x,\quad (x\overset{\ell}{\circ}y)\circ y=x,\quad x\overset{r}{\circ}(x\circ y)=y,\quad x\circ(x\overset{r}{\circ}y)=y \tag{1}$$

is called a *quasigroup*; the operation $(\circ)$ is *main*, $(\overset{\ell}{\circ})$, $(\overset{r}{\circ})$ are called *left* and *right divisions* of $(\circ)$. The operation $(\circ)$ is also called *invertible* because $(\overset{\ell}{\circ})$, $(\overset{r}{\circ})$ are its *left* and *right inverse* elements in the semigroups $(O_2;\underset{\ell}{\oplus})$ and $(O_2;\underset{r}{\oplus})$ respectively, where O_2 denotes the set of all binary operations defined on Q and

$$(f\underset{\ell}{\oplus}g)(x,y):=f(g(x,y),y),\qquad (f\underset{r}{\oplus}g)(x,y):=f(x,g(x,y)).$$

The set of all invertible binary operations defined on Q is denoted by Δ_2. Each inverse of an invertible operation is also invertible. All such operations are called parastrophes of $(\circ)$ and they are defined by

$$x_{1\sigma}\overset{\sigma}{\circ}x_{2\sigma}=x_{3\sigma}:\Leftrightarrow x_1\circ x_2=x_3,$$

where $\sigma\in S_3:=\{\iota,\ \ell,\ r,\ s,\ s\ell,\ sr\}$, $\ell:=(13)$, $r:=(23)$, $s:=(12)$. In particular, the left and right divisions of $(\circ)$ are its parastrophes. It is easy to verify equality $\overset{\sigma}{\left(\overset{\tau}{\circ}\right)}=\left(\overset{\sigma\tau}{\circ}\right)$ for all $\sigma,\tau\in S_3$, thus S_3 acts on the set Δ_2.

The stabilizer and the orbit of an invertible operation f under this action are called *parastrophic symmetry group* $\mathrm{Ps}(f)$ and *parastrophy orbit* $\mathrm{Po}(f)$ respectively. Consequently, $|\mathrm{Ps}(f)|\cdot|\mathrm{Po}(f)|=6$.

Let P be an arbitrary proposition in a class of quasigroups $\mathfrak{A}$. A proposition ${}^{\sigma}P$ is said to be a σ-parastrophe of P, if it can be obtained from P by replacing the main operation with its σ^{-1}-parastrophe.

Let ${}^{\sigma}\mathfrak{A}$ denote the class of all σ-parastrophes of quasigroups from $\mathfrak{A}$. A set of all pairwise parastrophic classes is called a *parastrophy orbit* of $\mathfrak{A}$ [10]:

$$\mathrm{Po}(\mathfrak{A})=\left\{{}^{\sigma}\mathfrak{A}\ \middle|\ \sigma\in S_3\right\}.$$

A parastrophy orbit of varieties is uniquely defined by one of its varieties.

Proposition 1. If quasigroup varieties coincide, then σ-parastrophes of these varieties also coincide.

Since a parastrophic orbit of varieties is the set of all parastrophes of one of them, then the following assertion is evident.

Corollary 1. A variety is totally symmetric, if it is an intersection of all varieties of a parastrophic orbit.

Theorem 1. [11] Let $\mathfrak{A}$ be a class of quasigroups, then a proposition P is true in $\mathfrak{A}$ if and only if ${}^{\sigma}P$ is true in ${}^{\sigma}\mathfrak{A}$ for all $\sigma\in S_3$.

Corollary 2. [11] Let P be true in a totally symmetric class $\mathfrak{A}$, then ${}^{\sigma}P$ is true in $\mathfrak{A}$ for all σ.

Corollary 3. [11] An identity $\omega=\upsilon$ defines a variety of quasigroups $\mathfrak{A}$ if and only if σ-parastrophe ${}^{\sigma}(\omega=\upsilon)$ of this identity defines the variety ${}^{\sigma}\mathfrak{A}$, where $\sigma\in S_3$.

A set of all parastrophes of A and all their finite intersections is called a bunch of the class A [11]. Therefore, a bunch of varieties is a parastrophically closed semilattice of varieties.

A quasigroup is called: a *LIP*, *RIP*, *MIP* quasigroup, if there exist transformations λ, ρ, μ called a *left, right*, *middle inversion mapping* such that for all x and y the respective equalities

$$\lambda(x)\cdot xy = y;\quad yx\cdot\rho(x) = y;\quad x\cdot y = \mu(y\cdot x)$$

are true. A quasigroup $(Q;\cdot)$ will be called: a middle CIP quasigroup, a left CIP quasigroup, a right CIP quasigroup, if there exist transformations ψ, ε, γ called a *middle*, *left*, *right inversion mapping* such that for all x and y the respective equalities

$$\psi(x)\cdot yx = y;\quad yx\cdot y = \varepsilon(x);\quad y\cdot xy = \gamma(x)$$

are true.

The concept of mirror quasigroups are introduced in [10]. A quasigroup $(Q;\cdot)$ is called: a *middle mirror IP quasigroup*, a *left mirror IP quasigroup*, a *right mirror IP quasigroup*, if there exists a transformation φ, δ, ξ called a *middle, left, right inversion mapping* such that for all x and y the respective equalities

$$\varphi(x)\cdot y = y\cdot x;\quad y\cdot yx = \delta(x);\quad xy\cdot y = \xi(x)$$

are true. A groupoid $(B;\cdot)$ is called an *isotope* of a groupoid $(A;\circ)$, if there are bijections α, β, γ from A to B such that the equality $\gamma(x\circ y) = \alpha(x)\cdot\beta(y)$ holds for all $x, y\in A$. The triple (α,β,γ) is called an *isotopism* between $(A;\circ)$ and $(B;\cdot)$; the bijections α, β, γ are called its *left, right* and *middle components*.

A quasigroup is called a *group isotope*, if it is isotopic to a group. If there exists a group $(Q;+,0)$ and bijections α, β and also an element a such that $\alpha 0 = \beta 0 = 0$ and

$$x\circ y = \alpha x + a + \beta y \tag{2}$$

for all x, y in Q, then the quadruble $(+,\alpha,\beta,a)$ is called a 0-canonical decomposition of the group isotope $(Q;\circ)$. In each group isotope, an arbitrary element 0 uniquely defines its 0-canonical decomposition [5].

A quasigroup $(Q;\circ)$ is called *linear*, if it is a group isotope and the coefficients of a canonical decomposition are automorphisms of the canonical decomposition group.

Theorem 2. [12] Each m-order quasigroup being linear over a cyclic group is isomorphic to exactly one quasigroup $(\mathbb{Z}_m;\circ)$, where $\mathbb{Z}_m$ is the ring modulo m, $x\circ y = ax + c + by$, a, b relatively prime to m, and c is a common factor of m and $a+b-1$.

Let $(+,\alpha,\beta,a)$ be a canonical decomposition of a group isotope $(Q;\circ)$. Then it is easy to see that all parastrophes of $(Q;\circ)$ have the following forms:

$$x\overset{\ell}{\circ}y = \alpha x + a + \beta y,\qquad x\overset{s}{\circ}y = \beta x + a + \alpha y,$$

$$x\overset{\ell}{\circ}y = \alpha^{-1}(x - a - \beta y),\qquad x\overset{s\ell}{\circ}y = \alpha^{-1}(-\beta x - a + y),$$

$$x\overset{r}{\circ}y = \beta^{-1}(-\alpha x - a + y),\qquad x\overset{sr}{\circ}y = \beta^{-1}(x - a - \alpha y).$$

Proposition 2. [5, Corollary 1] Let $(Q;+,0)$ be a group, α, β_1, β_2, β_3, β_4 be bijections of Q, besides $\alpha 0 = 0$ and let $\alpha(\beta_1 x + \beta_2 y) = \beta_3 u + \beta_4 v$ hold for all $x, y \in Q$. Then α is an automorphism of $(Q;+)$, if $u = x$, $v = y$ and it is an anti-automorphism of $(Q;+)$, if $u = y$, $v = x$.

Corollary 4. [4] If a group isotope $(Q;\cdot)$ satisfies an identity $\omega_1 x \cdot \omega_2 y = \omega_3 y \cdot \omega_4 x$, and the variables x, y are quadratic, then $(Q;\cdot)$ is isotopic to a commutative group.

Theorem 3. [7] Let $(Q;\cdot)$ be a group isotope and (2) be its canonical decomposition, then

1) $(Q;\cdot)$ is commutative if and only if $(Q;+)$ is abelian and $\beta = \alpha$;

2) $(Q;\cdot)$ is left symmetric if and only if $(Q;+)$ is abelian and $\beta = -\iota$;

3) $(Q;\cdot)$ is right symmetric if and only if $(Q;+)$ is abelian and $\alpha = -\iota$;

4) $(Q;\cdot)$ is totally symmetric if and only if $(Q;+)$ is abelian and $\beta = \alpha = -\iota$;

5) $(Q;\cdot)$ is semi-symmetric if and only if α is an anti-automorphism of $(Q;+)$, $\beta = \alpha^{-1}$, $\alpha^3 = -I_a^{-1}$, $\alpha a = -a$, where $I_a(x) = -a + x + a$;

6) $(Q;\cdot)$ is asymmetric if and only if $(Q;+)$ is not abelian or $-\iota \neq \alpha \neq \beta \neq -\iota$ and at least one of the following conditions is true: α is not an anti-automorphism, $\beta \neq \alpha^{-1}$, $\alpha^3 \neq -I_a^{-1}$, $\alpha a \neq -a$.

Theorem 4. [2] A left linear quasigroup $(Q;\cdot)$ over a loop $(Q;+)$, where $x \cdot y = a + \alpha x + \beta y$, is a CI-quasigroup relative to the permutation γ, where $\gamma 0 = 0$, if and only if $a + \alpha a = 0$, $\beta = \alpha^{-1}$, $\alpha^3 x + \gamma x = 0$ for all $x \in Q$ and $(Q;+)$ is a CI-loop.

2. A group isotope with inverse property.

Theorem 5. [7] The parastrophic orbit of IP-quasigroups consists of three varieties: middle $\mathfrak{I}$, left ${}^{\ell}\mathfrak{I}$ and right ${}^{r}\mathfrak{I}$ IP-quasigroups respectively.

$\mathfrak{I} = {}^{s}\mathfrak{I}$	${}^{\ell}\mathfrak{I} = {}^{sr}\mathfrak{I}$	${}^{r}\mathfrak{I} = {}^{s\ell}\mathfrak{I}$
$(\exists\mu)$ $x \cdot y = \mu(y \cdot x)$	$(\exists\lambda)$ $\lambda(x) \cdot xy = y$;	$(\exists\rho)$ $yx \cdot \rho(x) = y$;
$yx = z \cdot (xy \overset{\ell}{\cdot} z)$	$(z \overset{\ell}{\cdot} xz) \cdot xy = y$	$yx \cdot (zx \overset{r}{\cdot} z) = y$

In [7] the group isotopes (in the case $\rho 0 = 0$ and $\lambda 0 = 0$) in each of varieties of the inverse property quasigroups are described. A more general case is considered in the following theorem.

Theorem 6. Let $(Q;\circ)$ be a group isotope and (2) be its canonical decomposition, then

1) $(Q;\circ)$ is an RIP quasigroup with the inversion mapping ρ if and only if α is an involutive automorphism of $(Q;+)$. The inversion mapping in an MIP group isotope with (2) is $\rho(x) = \beta^{-1}(-a - \alpha\beta x - \alpha a)$;

2) $(Q;\circ)$ is an LIP quasigroup with the inversion mapping λ if and only if β is an involutive automorphism of $(Q;+)$. The inversion mapping in an LIP group isotope with (2) is $\lambda(x) = \alpha^{-1}(-\beta a - \beta\alpha x - a)$;

3) $(Q;\circ)$ is an MIP quasigroup with the inversion mapping μ if and only if there exists an automorphism θ such that $\mu(x) = \theta x + c$, $\theta^2 = I_c^{-1}$, $\alpha = \theta\beta$, where $c := -\theta a + a$.

Proof.

1) Let a group isotope $(Q;\circ)$ be an RIP quasigroup with an inversion mapping ρ, i.e., the identity $(y\circ x)\circ\rho x = y$ holds. Using the canonical decomposition (2) of $(Q;\circ)$, we have

$$\alpha(\alpha y + a + \beta x) + a + \beta\rho(x) = y\,.$$

Therefore, $\alpha(\alpha y + a + \beta x) = y - \beta\rho(x) - a$. Proposition 1 implies that α is an automorphism of $(Q;+,0)$ then

$$\alpha^2 y + \alpha a + \alpha\beta x = y - \beta\rho(x) - a\,. \tag{3}$$

If $x = y = 0$, we obtain $\alpha a = -\beta\rho 0 - a$ and if $x = 0$, we have

$$\alpha^2 y + \alpha a = y - \beta\rho 0 - a\,.$$

Consequently, $\alpha^2 = \iota$. Putting the obtained relation into (3), we have

$$y + \alpha a + \alpha\beta x = y - \beta\rho(x) - a.$$

Reducing y on the left in the equality and adding a on the right: $\alpha a + \alpha\beta x + a = -\beta\rho(x)$, i.e., $\beta\rho x = -a - \alpha\beta x - \alpha a$. Therefrom,

$$\rho(x) = \beta^{-1}(-a - \alpha\beta x - \alpha a)\,. \tag{4}$$

Conversely, let $(Q;\circ)$ be a group isotope with the canonical decomposition (2) satisfying (4) then

$$(y\circ x)\circ\rho(x) \overset{(2)}{=} \alpha(\alpha y + a + \beta x) + a + \beta\rho(x) = \alpha^2 y + \alpha a + \alpha\beta x + a + \beta\rho(x) =$$
$$\overset{(4)}{=} y + \alpha a + \alpha\beta x + a + \beta\beta^{-1}(-a - \alpha\beta x - \alpha a) =$$
$$= y + \alpha a + \alpha\beta x + a - a - \alpha\beta x - \alpha a = y.$$

Thus, the group isotope $(Q;\circ)$ has the right inverse property.

2) Let a group isotope $(Q;\circ)$ be an LIP quasigroup with an inversion mapping λ, i.e., the identity $\lambda(x)\circ(x\circ y) = y$ holds. Using the canonical decomposition (2) of $(Q;\circ)$, we have: $\alpha\lambda(x) + a + \beta(\alpha x + a + \beta y) = y$, therefrom

$$\beta(\alpha x + a + \beta y) = -a - \alpha\lambda(x) + y.$$

Proposition 1 implies that β is an automorphism of $(Q;+,0)$, consequently

$$\beta\alpha x + \beta a + \beta^2 y = -a - \alpha\lambda(x) + y. \tag{5}$$

If $x = y = 0$, then $\beta a = -a - \alpha\lambda 0$; if $x = 0$, then $\beta a + \beta^2 y = -a - \alpha\lambda 0 + y$, i.e., $\beta^2 = \iota$.

Putting the obtained relations into (5): $\beta\alpha x + \beta a + y = -a - \alpha\lambda(x) + y$. Reducing y on the right, we obtain:

$$\beta\alpha x + \beta a = -a - \alpha\lambda(x), \text{ i.e., } \alpha\lambda x = -\beta a - \beta\alpha x - a.$$

Hence,

$$\lambda(x) = \alpha^{-1}(-\beta a - \beta\alpha x - a)\,. \tag{6}$$

Conversely, let $(Q;\circ)$ be a group isotope with the canonical decomposition (2). Let λ be defined by (6), then

$$\lambda(x)\circ(x\circ y)\overset{(2)}{=}\alpha\lambda(x)+a+\beta(\alpha x+a+\beta y)=\alpha\lambda(x)+a+\beta\alpha x+\beta a+\beta^2 y=$$

$$\overset{(6)}{=}\alpha\alpha^{-1}(-\beta a-\beta\alpha x-a)+a+\beta\alpha x-a+\beta^2 y=$$

$$=-\beta a-\beta\alpha x-a+a+\beta\alpha x+\beta a+\beta^2 y=y.$$

Thus, the group isotope $(Q;\circ)$ has the left inverse property.

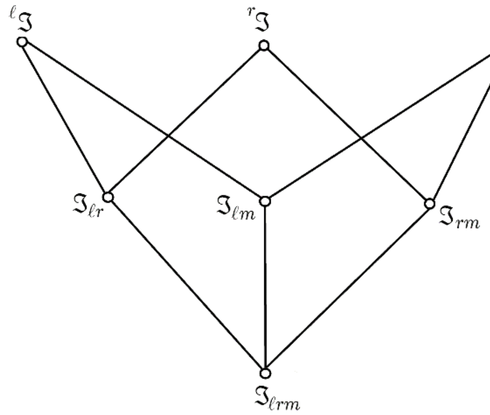
The item 3) has been proved in [7].

Proposition 3 [7]. The bunch of the varieties of IP quasigroups consists of the following varieties:

1) The parastrophic orbit of one-sided IP quasigroups $\mathrm{Po}(\mathfrak{I})=\{\mathfrak{I},{}^{\ell}\mathfrak{I},{}^{r}\mathfrak{I}\}$;

2) The parastrophic orbit of two-sided IP quasigroups $\mathrm{Po}(\mathfrak{I})=\{\mathfrak{I}_{\ell m},\mathfrak{I}_{\ell r},\mathfrak{I}_{rm}\}$;

3) The parastrophic orbit of three-sided IP quasigroups $\mathfrak{I}_{\ell rm}=\mathfrak{I}\cap{}^{\ell}\mathfrak{I}\cap{}^{r}\mathfrak{I}$



Consider linear isotopes of cyclic groups. According to Theorem 2, each of these quasigroups is isomorphic to a quasigroup whose operation is a polynomial in the ring $(\mathbb{Z}_m,+,\cdot)$ modulo m.

Corollary 5. Let $(\mathbb{Z}_m;\circ)$ be a group isotope with the canonical decomposition $x\circ y=ax+c+by$, where c is a common factor of m and $a+b-1$, then:

1) $(\mathbb{Z}_m;\circ)$ is a right IP quasigroup if and only if $a^2=1$. The inversion mapping is $\rho(x)=-b^{-1}c-ax-ab^{-1}c$;

2) $(\mathbb{Z}_m;\circ)$ is a left IP quasigroup if and only if $b^2=1$. The inversion mapping is $\lambda(x)=-a^{-1}bc-bx-a^{-1}c$;

3) $(\mathbb{Z}_m;\circ)$ is a middle IP quasigroup if and only if $a^2=b^2$. The inversion mapping is $\mu(x)=a^{-1}bx-a^{-1}bc+c$.

Proof.

The proof of the item 1) and the item 2) immediately follows from Theorem 2 and Theorem 6.

Consider the item 3). Let a group isotope $(\mathbb{Z}_m;\circ)$ be an MIP quasigroup with an inversion mapping μ. According to Theorem 6, $(\mathbb{Z}_m;\circ)$ is a middle IP quasigroup with the inversion mapping μ if and only if there exist k such that $\mu(x)=kx+d$, $k^2=1$, $a=kb$, where $d:=-kc+c$. Since $k=ab^{-1}$, then $k^2=1$ is equivalent to $a^2=b^2$. Then

$$\mu(x)=kx+d=kx-kc+c=ab^{-1}x-ab^{-1}c+c.$$

3. A group isotope with cross inverse property.

Theorem 7. [10] The parastrophic orbit of the CIP quasigroups consists of three varieties: middle A, left ${}^{\ell}$A and right ^{r}A of CIP-quasigroups respectively.

$A={}^{s}A$	${}^{\ell}A={}^{sr}A$	${}^{r}A={}^{s\ell}A$
$(\exists\psi)\ \psi(x)\cdot yx=y$	$(\exists\varepsilon)\ xy\cdot x=\varepsilon(y)$	$(\exists\gamma)\ x\cdot yx=\gamma(y)$
$xy\cdot(xz\overset{r}{\cdot}z)=y$	$yx\cdot y=zx\cdot z$	$y\cdot xy=z\cdot xz$

Theorem 8. Let $(Q;\circ)$ be a group isotope and (2) be its canonical decomposition, then:

1) $(Q;\circ)$ is an MCIP quasigroup with the inversion mapping ψ if and only if α is an anti-automorphism of $(Q;+)$ and $\beta=\alpha^{-1}$. The inversion mapping in an MCIP group isotope is $\psi(x)=-\alpha^{-2}a-\alpha^{-3}x-\alpha^{-1}a$;

2) $(Q;\circ)$ is an LCIP quasigroup with the inversion mapping ε if and only if α is an anti-automorphism of $(Q;+)$ and $\beta=I_aJ\alpha^2$. The inversion mapping in an LCIP group isotope is $\varepsilon(x)=\alpha\beta x+\alpha a+a$.

3) $(Q;\circ)$ is an RCIP quasigroup with the inversion mapping γ if and only if β an anti-automorphism of $(Q;+)$ and $\alpha=I_a^{-1}J\beta^2$. The inversion mapping in an RCIP group isotope is $\gamma(x)=a+\beta a+\beta\alpha x$.

Proof.

1) Let a group isotope $(Q;\circ)$ be an MCIP quasigroup, namely the identity $\psi(x)\circ(y\circ x)=y$ holds. Using the canonical decomposition (2), we have:

$$\alpha\psi(x)+a+\beta(\alpha y+a+\beta x)=y,$$

therefore, $\beta(\alpha y+a+\beta x)=-a-\alpha\psi(x)+y.$ Proposition 1 implies that β, α are anti-automorphisms of $(Q;+,0)$, then

$$\beta^2x+\beta a+\beta\alpha y=-a-\alpha\psi(x)+y. \tag{7}$$

If $x=y=0$, then $\beta a=-a-\alpha\psi 0$. If $x=0$, then $\beta a+\beta\alpha y=-a-\alpha\psi 0+y$, i.e., $\beta a+\beta\alpha y=\beta a+y$. From here, $\beta\alpha=\iota$, i.e., $\beta=\alpha^{-1}$. Putting these relations into (7), we get:

$$(\alpha^{-1})^2x+\alpha^{-1}a+y=-a-\alpha\psi(x)+y.$$

Reducing y, we obtain: $\alpha^{-2}x+\alpha^{-1}a=-a-\alpha\psi(x).$ Therefrom,

$$-\alpha\psi(x)=a+\alpha^{-2}x+\alpha^{-1}a. \tag{8}$$

Since $-\alpha$ is an automorphism of $(Q;+,0)$, then

$$\psi(x)=-\alpha^{-1}a-\alpha^{-3}x-\alpha^{-2}a. \tag{9}$$

Conversely, let $(Q;\circ)$ be a group isotope with the canonical decomposition (2) besides α be an anti-automorphism of $(Q;+,0)$, $\beta=\alpha^{-1}$ and let ψ be defined by (9):

$$\psi(x)\circ(y\circ x)\overset{(2)}{=}\alpha\psi(x)+a+\alpha^{-1}(\alpha y+a+\alpha^{-1}x)=\alpha\psi x+a+\alpha^{-2}x+\alpha^{-1}a+y=$$

$$\overset{(9)}{=}\alpha(-\alpha^{-1}a-\alpha^{-3}x-\alpha^{-2}a)+a+\alpha^{-2}x+\alpha^{-1}a+y=$$

$$=-\alpha^{-1}a-\alpha^{-2}x-a+a+\alpha^{-2}x+\alpha^{-1}a+y=y.$$

Thus, the group isotope $(Q;\circ)$ is an MCIP quasigroup.

2) Let a group isotope $(Q;\circ)$ be a LCIP quasigroup, i.e., it is defined by the identity $(x\circ y)\circ x=\varepsilon(y)$ for some ε. Using the canonical decomposition (2) of the operation $(\circ)$, the identity can be written as $\alpha(\alpha x+a+\beta y)+a+\beta x=\varepsilon(y)$. Hence, $\alpha(\alpha x+a+\beta y)=\varepsilon(y)-\beta x-a.$ Proposition 1 implies that α is an anti-automorphism of $(Q;+,0)$, therefore

$$\alpha\beta y+\alpha a+\alpha^2x=\varepsilon(y)-\beta x-a. \qquad (10)$$

If $x=y=0,$ then $\alpha a=\varepsilon 0-a$. For $x=0$ we have $\alpha\beta y+\alpha a=\varepsilon(y)-a.$ Thus, $\varepsilon(y)=\alpha\beta y+\alpha a+a$. Putting the obtained relation into the last equality, we get:

$$\alpha\beta y+\alpha a+\alpha^2x=\alpha\beta y+\alpha a+a-\beta x-a\,.$$

Reducing $\alpha\beta y+\alpha a$, we obtain $\alpha^2x=a-\beta x-a$. Thus,

$$\beta x=-a-\alpha^2x+a,\ \text{ i.e., }\ \beta=I_aJ\alpha^2.$$

Vice versa, let $(Q;\circ)$ be a group isotope with the canonical decomposition (2) and also $\beta=I_aJ\alpha^2.$ We'll show that $(Q;\circ)$ is an LCIP quasigroup with the inversion mapping $\varepsilon(y):=\alpha\beta y+\alpha a+a$, i.e., the identity $(x\circ y)\circ x=\varepsilon(y)$ holds. Indeed,

$$(x\circ y)\circ x\overset{(2)}{=}\alpha(\alpha x+a+\beta y)+a+\beta x=\alpha\beta y+\alpha a+\alpha^2x+a+I_aJ\alpha^2x=$$

$$=\alpha\beta y+\alpha a+\alpha^2x+a-a-\alpha^2x+a=\alpha\beta y+\alpha a+a=\varepsilon(y).$$

Thus, the group isotope $(Q;\circ)$ is an LCIP quasigroup.

3) Let a group isotope $(Q;\circ)$ be an RCIP quasigroup, i.e., a quasigroup which satisfies the identity $y\circ(x\circ y)=\gamma(x)$. Using its canonical decomposition (2), the identity can be written as:

$\alpha y+a+\beta(\alpha x+a+\beta y)=\gamma(x),$

therefore $\beta(\alpha x+a+\beta y)=-a-\alpha y+\gamma(x).$ By Proposition 1 the transformation β is an anti-automorphism of the group $(Q;+,0)$, hence

$$\beta^2y+\beta a+\beta\alpha x=-a-\alpha y+\gamma(x). \qquad (11)$$

If $x=y=0,$ the identity is $\alpha a=-a+\gamma 0$ and if $y=0$, it is $\beta a+\beta\alpha x=-a+\gamma(x),$ i.e., $\gamma(x)=a+\beta a+\beta\alpha x$. Putting the obtained relations into (11), we get:

$$\beta^2y+\beta a+\beta\alpha x=-a-\alpha y+a+\beta a+\beta\alpha x\,.$$

Reducing $\beta a+\beta\alpha x$, we have $\beta^2y=-a-\alpha y+a$. Thus,

$$\alpha y = a - \beta^2 y - a, \text{ i.e., } \alpha = I_a^{-1} J \beta^2.$$

Vice versa, let $(Q;\circ)$ be a group isotope with the canonical decomposition (2), $\alpha = I_a^{-1} J\beta^2$ holds. We'll show that $(Q;\circ)$ is an RCIP quasigroup with the inversion mapping $\gamma(x) := a + \beta a + \beta\alpha x$; in other words, the identity $y \circ (x \circ y) = \gamma(x)$ is true:

$$y \circ (x \circ y) \overset{(2)}{=} \alpha y + a + \beta(\alpha x + a + \beta y) = \alpha y + a + \beta^2 y + \beta a + \beta\alpha x =$$

$$= I_a^{-1} J\beta^2 y + a + \beta^2 y + \beta a + \beta\alpha x = a - \beta^2 y - a + a + \beta^2 y + \beta a + \beta\alpha x =$$

$$= a + \beta a + \beta\alpha x = \gamma(x).$$

Hence, the group isotope $(Q;\circ)$ is an RCIP quasigroup.

Theorem 9. If a group isotope has two of the following properties: LCIP, RCIP, MCIP, then it also satisfies the third one.

Proof. Let $(Q;\circ)$ be a group isotope and (2) be its canonical decomposition. By Theorem 6, it is enough to prove that any two equalities

$$\beta = JI_a\alpha^2, \quad \alpha = JI_a^{-1}\beta^2, \quad \beta = \alpha^{-1}$$

imply the third one.

Let the first two equalities hold. Replace the first equality with the second one:

$$\alpha = JI_a^{-1}\beta^2 = JI_a^{-1}(JI_a\alpha^2)\cdot(JI_a\alpha^2) = \alpha^2(JI_a)\alpha^2.$$

Therefore, $\alpha^{-1} = JI_a\alpha^2 = \beta$, that is, the third equality is true.

Let the first and the third equalities hold. Substituting the third equality for the first one, we obtain: $\alpha^{-1} = JI_a\alpha^2$, i.e., $JI_a = \alpha^{-3}$.

Consequently, $JI_a^{-1}\beta^2 = JI_a^{-1}\beta\cdot\beta = JI_a^{-1}(JI_a\alpha^2)(JI_a\alpha^2) = \alpha^2(JI_a)\alpha^2 = \alpha^2\alpha^{-3}\alpha^2 = \alpha$ that is, the second equality holds.

Let the second and the third equalities hold. Replacing the third equality with the second one, we have $\beta^{-1} = JI_a^{-1}\beta^2$, i.e., $JI_a^{-1} = \beta^{-3}$.

Therefrom, $JI_a\alpha^2 = JI_a\alpha\cdot\alpha = JI_a(JI_a^{-1}\beta^2)(JI_a^{-1}\beta^2) = \beta^2(JI_a^{-1})\beta^2 = \beta^2\beta^{-3}\beta^2 = \beta$.

Thus, the first equality holds.

In [12] it has been proved that each linear isotope of a cyclic group is isomorphic to a quasigroup which is defined on the ring modulo m (see Theorem 2). That is why the next corollary has been proved for all quasigroups. All these quasigroups are in fact linear over the finite cyclic groups.

Corollary 6. Let $(\mathbb{Z}_m;+,\cdot)$ be the ring modulo m and $(\mathbb{Z}_m;\circ)$ be a group isotope with the canonical decomposition $x \circ y = ax + c + by$, where c is a common factor of m and $a + b - 1$, then:

1) $(\mathbb{Z}_m;\circ)$ is a middle CIP quasigroup if and only if $ab = 1$ in $\mathbb{Z}_m$. The inversion mapping ψ of the MCIP quasigroup is $\psi(x) = -b^2 - b^3x - bc$;

2) $(\mathbb{Z}_m;\circ)$ is a left CIP quasigroup if and only if $a^2 + b = 0$ in $\mathbb{Z}_m$. The inversion mapping ε of the LCIP quasigroup is $\varepsilon(x) = abx + ac + c$;

3) $(\mathbb{Z}_m;\circ)$ is a right CIP quasigroup if and only if $a + b^2 = 0$ in $\mathbb{Z}_m$. The inversion mapping γ of the RCIP quasigroup is $\gamma(x) = c + bc + bax$.

Proof. The proof immediately follows from Theorem 2 and Theorem 8.

Note, some properties of middle CIP quasigroups which are linear isotopes of cyclic groups are obtained in [13].

Example 1.

The quasigroup $(\mathbb{Z}_5;\circ)$, where $x \circ y := 2x + 2 + 3y$, is a middle CIP quasigroup and it is neither left CIP quasigroup nor right CIP quasigroup.

Indeed according to Corollary 6, it is a middle CIP quasigroup, since $2 \cdot 3 = 1$ in $\mathbb{Z}_5$. But it is neither left CIP quasigroup nor right CIP quasigroup because $2^2 + 3 = 4 + 3 = 2 \neq 0$ and $2 + 3^2 = 2 + 4 = 1 \neq 0$ in $\mathbb{Z}_5$.

By Proposition 1, the variety with a three-sided cross inverse property is totally symmetric. We demonstrate this property by using example 2.

Example 2.

Consider a quasigroup $(\mathbb{Z}_7;\circ)$, where $x \circ y := 3x + 2 + 5y$, over the field $\mathbb{Z}_7$. By Corollary 6, the quasigroup $(\mathbb{Z}_7;\circ)$ belongs to each of the varieties A, ${}^{\ell}\mathrm{A}$, ${}^{r}\mathrm{A}$ because

$$3 \cdot 5 = 1, \qquad 3^2 + 5 = 2 + 5 = 0, \qquad 3 + 5^2 = 3 + 4 = 0$$

in $\mathbb{Z}_7$. But the middle ψ, left ε and right γ inversion mappings are different:

$$\psi(x) := x + 4, \qquad \varepsilon(x) := x + 1, \qquad \gamma(x) := x + 5.$$

In addition, we check the corresponding identities:

$$(x \circ y) \circ \psi(x) = 3(3x + 2 + 5y) + 2 + 5(x + 4) = 2x + 6 + y + 2 + 5x + 6 = y,$$

$$(y \circ x) \circ y = 3(3y + 2 + 5x) + 2 + 5y = 2y + 6 + x + 2 + 5y = x + 1 = \varepsilon(x),$$

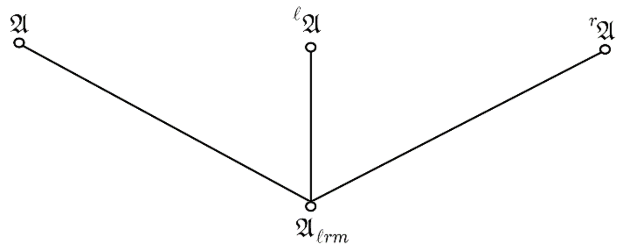
$$y \circ (x \circ y) = 3y + 2 + 5(3x + 2 + 5y) = 3y + 2 + x + 3 + 4y = x + 5 = \gamma(x).$$

Since the equations that characterize the varieties of CIP quasigroups hold, the quasigroup $(\mathbb{Z}_7;\circ)$ is totally symmetric.

Proposition 4. The bunch of varieties of CIP quasigroups consists of the following varieties:

1) The parastrophic orbit of one-sided CIP quasigroups $\mathrm{Po}(\mathrm{A}) = \{\mathrm{A}, {}^{\ell}\mathrm{A}, {}^{r}\mathrm{A}\}$;

2) The parastrophic orbit of three-sided CIP quasigroups $\mathrm{A}_{\ell rm} = \mathrm{A} \cap {}^{\ell}\mathrm{A} \cap {}^{r}\mathrm{A}$.



Proof. The proof of the item 1) follows from Theorem 7 and Example 1; the proof of the item 2) follows from Proposition 1 and Example 2.

Theorem 10. [10] The parastrophic orbit of mirror quasigroups consists of three varieties: middle M, left ${}^{\ell}\mathrm{M}$ and right ${}^{r}\mathrm{M}$ of mirror quasigroups respectively.

$\mathrm{M} = {}^{s}\mathrm{M}$	${}^{\ell}\mathrm{M} = {}^{sr}\mathrm{M}$	${}^{r}\mathrm{M} = {}^{s\ell}\mathrm{M}$
$(\exists\varphi)\ \varphi(x) \cdot y = y \cdot x$	$(\exists\delta)\ y \cdot yx = \delta(x)$	$(\exists\xi)\ xy \cdot y = \xi(x)$

$(zx \overset{\ell}{\cdot} z)\cdot y = yx$	$y\cdot yx = z\cdot zx$	$xy\cdot y = xz\cdot z$

Theorem 11. In the variety of group isotopes the following assertions are true:

1) the subvariety of MMIP quasigroups coincides with the subvariety of commutative quasigroups;

2) the subvariety of LMIP quasigroups coincides with the subvariety of left symmetric quasigroups;

3) the subvariety of RMIP quasigroups coincides with the subvariety of right symmetric quasigroups;

4) the subvariety of left, right and middle mirror quasigroups coincides with the subvariety of totally symmetric quasigroups.

Proof. Suppose $(Q;\cdot)$ is a group isotope with the canonical decomposition (2):

1) Let $(Q;\cdot)$ be a middle mirror quasigroup, that is, the identity $\varphi(x)\cdot y = y\cdot x$ holds. Taking into account (2), the identity can be written as

$$\alpha\varphi(x)+a+\beta y = \alpha y + a + \beta x.$$

By Corollary 4, the group $(Q;+)$ is abelian. If $x=y=0,$ we obtain $\alpha\varphi 0 = 0$. If $x=0$, then $a\varphi 0+\beta y = \alpha y$ and thus $\beta=\alpha$. According to Theorem 3, $(Q;\cdot)$ is a commutative quasigroup.

Conversely, let $(Q;\cdot)$ be commutative, then the identity $\varphi(x)\cdot y = y\cdot x$ with $\varphi=\iota$ is true. Therefore, $(Q;\cdot)$ is a middle mirror quasigroup. Thus, the subvariety of MMIP quasigroups coincides with the subvariety of commutative quasigroups.

2) Let $(Q;\cdot)$ be a LMIP quasigroup, that is, the identity $y\cdot yx = \delta(x)$ holds. Using its canonical decomposition (2), we have: $\alpha y + a + \beta(\alpha y + a + \beta x) = \delta(x)$.

Replacing $\alpha y + a$ with y, we obtain

$$y+\beta\left(y+\beta x\right)=\delta\left(x\right),\quad \text{i.e.,}\quad \beta\left(y+\beta x\right) = -y+\delta\left(x\right).$$

Proposition 1 implies that β is an automorphism of $(Q;+,0)$. If $x=0$, we have $\beta y = -y$, i.e., $\beta=-\iota$.

Since α is an automorphism and an anti-automorphism, then $(Q;+)$ is an abelian group.

Thus, $\varphi(x)=\beta^2 x = x.$ According to Theorem 3, if $\varphi=\iota$, then $(Q;\cdot)$ is left symmetric quasigroup.

Vice versa, let $(Q;\cdot)$ be a left symmetric quasigroup, then identity $y\cdot yx = \delta(x)$ with $\delta=\iota$ is true. Therefore, $(Q;\cdot)$ is a left mirror quasigroup. Thus, the subvariety of LMIP quasigroups coincides with the subvariety of left symmetric quasigroups.

3) Let a group isotope $(Q;\cdot)$ be a RMIP quasigroup, which defines by the identity $xy\cdot y = \xi(x)$. Using its canonical decomposition (2), we have: $\alpha(\alpha x + a + \beta y) + a + \beta y = \xi(x)$.

Replacing $a+\beta y$ with y, we obtain

$$\alpha\left(\alpha x + y\right) + y = \xi\left(x\right),\quad \text{i.e.,}\quad \alpha\left(\alpha x + y\right) = \xi\left(x\right) - y.$$

Proposition 1 implies that α is an automorphism of (Q; +, 0). If $x=0$, we have $\alpha y = -y$, i.e., $\alpha = -\iota$.

Since α is an automorphism and an anti-automorphism, then $(Q;+)$ is an abelian group.

Thus, $\varphi(x) = \alpha^2 x = x$. According to Theorem 3, if $\xi = \iota$, then $(Q;\cdot)$ is right symmetric quasigroup.
Vice versa, let $(Q;\cdot)$ be right symmetric quasigroup, then identity $xy \cdot y = \xi(x)$ with $\xi = \iota$ is true. Therefore, $(Q;\cdot)$ is a right mirror quasigroup. Thus, the subvariety of RMIP quasigroups coincides with the subvariety of right symmetric quasigroups.

The proof item 4) follows from Corollary 1 and Theorem 3.

Corollary 7. Let $(Q;\cdot)$ be a group isotope and (2) be its canonical decomposition, then:

1) $(Q;\cdot)$ is an MMIP quasigroup with the inversion mapping φ if and only if $(Q;+)$ is abelian and $\beta = \alpha$, $\varphi = \iota$;

2) $(Q;\cdot)$ is an LMIP quasigroup with the inversion mapping δ if and only if $(Q;+)$ is abelian and $\beta = -\iota$, $\delta = \iota$;

3) $(Q;\cdot)$ is an RMIP quasigroup with the inversion mapping ξ if and only if $(Q;+)$ is abelian and $\alpha = -\iota$, $\xi = \iota$.

Proof. The proof follows from Theorem 3 and Theorem 12.

Example 3.

Consider examples of quasigroups that are middle, left, and right mirror quasigroups. By corollary 7, we get:

1) the quasigroup $(\mathbb{Z}_7;\cdot)$, where $x \cdot y = 4x + 2 + 4y$ over the field $\mathbb{Z}_7$ belongs to the variety $\mathfrak{M}$ because $\alpha = \beta = 4$;

2) the quasigroup $(\mathbb{Z}_5;*)$, where $x * y = 5x + 3 + 4y$ over the field $\mathbb{Z}_5$ belongs to the variety ${}^{\ell}\mathfrak{M}$ because $\beta = 4 = -\iota$;

3) the quasigroup $(\mathbb{Z}_9;\circ)$, where $x \circ y = 8x + 1 + 3y$ over the ring $\mathbb{Z}_9$ belongs to the variety ${}^{r}\mathfrak{M}$ because $\alpha = 8 = -\iota$.

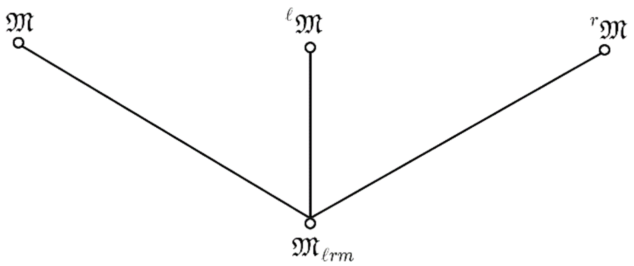
Corollary 8. If a group isotope has two of the properties: LMIP, RMIP, MMIP, then it also satisfies the third one.

Proof. The proof immediately follows from Theorem 10.

Proposition 5. The bunch of varieties of mirror quasigroups consists of the following varieties:

1) The parastrophic orbit of one-sided mirror quasigroups $Po(\mathfrak{M}) = \{\mathfrak{M}, {}^{\ell}\mathfrak{M}, {}^{r}\mathfrak{M}\}$;

2) The parastrophic orbit of three-sided mirror quasigroups $\mathfrak{M}_{\ell rm} = \mathfrak{M} \cap {}^{\ell}\mathfrak{M} \cap {}^{r}\mathfrak{M}$.



Proof. The proof of the item 1) follows from Theorem 9 and Example 3; the proof of the item 2) follows from Proposition 1.

Conclusion.

In this article, the criteria according to which we establish the affiliation of group isotopes to the varieties of IP quasigroups, CIP quasigroups and mirror quasigroups have been found. Bunches of group isotopes of IP quasigroups, CIP quasigroups, mirror quasigroups have been investigated. The corresponding semilattices have been constructed.

A classification of group isotopes according to inverse properties of their elements have been obtained. The classification is presented in the following table:

Variety	A group isotope $(Q;\cdot)$	Conditions of its canonical decomposition (2)
$\mathfrak{I}$	MIP	$\alpha=\theta\beta$, $\mu(x)=\theta x-\theta a+a$, $\theta^2=I_c^{-1}$, $c:=-\theta a+a$
$^{\ell}\mathfrak{I}$	LIP	$\beta^2=\iota$, $\lambda(x)=\alpha^{-1}(-\beta a-\beta\alpha x-a)$
$^{r}\mathfrak{I}$	RIP	$\alpha^2=\iota$, $\rho(x)=\beta^{-1}(-a-\alpha\beta x-\alpha a)$
A	MCIP	$\beta=\alpha^{-1}$, $\psi(x)=-\alpha^{-2}a-\alpha^{-3}x-\alpha^{-1}a$
$^{\ell}$A	LCIP	$\beta=I_aJ\alpha^2$, $\varepsilon(x)=\alpha\beta x+\alpha a+a$
^{r}A	RCIP	$\alpha=I_a^{-1}J\beta^2$, $\gamma(x)=a+\beta a+\beta\alpha x$
M	MMIP	$(Q;+)$ is abelian and $\beta=\alpha$, $\varphi=\iota$
$^{\ell}$M	LMIP	$(Q;+)$ is abelian and $\beta=-\iota$, $\delta=\iota$
^{r}M	RMIP	$(Q;+)$ is abelian and $\alpha=-\iota$, $\xi=\iota$

It has been proved that in the class of group isotopes, the subvarieties of middle, left and right mirror quasigroups coincide with the subvarieties of commutative, left-symmetric and right-symmetric quasigroups respectively.

It has been proved that two-sided CIP and two-sided mirror quasigroups do not exist. The question of constructing bunches of CIP and mirror quasigroups may be the subject of future research.

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КЛАСИФІКАЦІЯ ГРУПОВИХ ІЗОТОПІВ ЗГІДНО ЇХ ВЛАСТИВОСТЕЙ ОБОРОТНОСТІ

Співпадіння множин трансляцій однакових напрямків у квазігрупі визначає дев'ять многовидів: многовиди IP, CIP та дзеркальні квазігрупи [9]. Вивчено їх перетин із многовидом групових ізотопів. Зокрема, доведено, що у многовиді групових ізотопів підмноговиди середньої, лівої та правої дзеркальних квазігруп співпадають з підмноговидами комутативних, лівих та правих симетричних квазігруп відповідно.

Ключові слова: *квазігрупа, тотожність, многовид, парастроф, груповий ізотоп.*

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