https://doi.org/10.15407/apmm2020.18.150-161

УДК 512.548

F. M. Sokhatsky[1]✉, A. V. Tarasevych[2]

ON TERNARY QUASIGROUP QUADRATIC IDENTITIES OF THE SMALL LENGTH

In this article, it has been proved that each quadratic identity of the lengths one, two, three is parastrophically primarily equivalent to at least one of the given identities. The identities of the length three have been analyzed in the class of universal loops, i.e., quasigroups in which every element is neutral. It has been proved that there are five non-equivalent identities. The first identity defines the class of all universal loops, the second one defines the variety of the boolean skeins and the other three identities define three parastrophic varieties whose operations are repetition-free compositions of binary commutative middle loops.
.

Key words: ternary quasigroup, quadratic equation, universal loop, Steiner quasigroup, identity, middle loop.

Introduction. This article is the continuation of the articles [3, 4] and all necessary concepts and results can be found there.

In [3], the quadratic generalized functional equations of the length three on invertible functions (i.e., quasigroup operations) are studied: general solutions of each element from the family of pairwise parastrophically primarily non-equivalent equations have been found; a full proof of the classification theorem is given.

A quasigroup functional equation, whose functional variables are parastrophic, is an identity in the variety of quasigroups. A ternary quasigroup in which each element is neutral, is called universal loop. For example, a ternary Steiner quasigroup is exactly a totally symmetric universal loop.

In this article, the solutions of all representatives of the classification in the variety of all universal loops are given (Theorem 7). Using the obtained results, it is proved that each ternary quasigroup quadratic identity of the length three is equivalent to exactly one of the given identities (Theorem 3). The first identity defines the class of all universal loops, the second one defines the variety of the boolean skeins [2] and the other three identities define three parastrophic varieties whose operations are repetition-free compositions of binary commutative middle loops (Theorem 4). The identities of the lengths one and two are parastrophically primarily equivalent to at least one of the identities given in Corollaries 1 and 2.

1. Preliminaries.

An algebra $(Q;\circ,\overset{\ell}{\circ},\overset{r}{\circ})$ is called a *quasigroup* if the following identities are true:

$$(x\overset{\ell}{\circ}y)\circ y=x\,,\quad (x\circ y)\overset{\ell}{\circ}y=x\,,\quad x\circ(x\overset{r}{\circ}y)=y\,,\quad x\overset{r}{\circ}(x\circ y)=y\,.$$

An element 0 is called *left*, *right*, *middle neutral* for an invertible operation $(\circ)$, if the respective identity is true:

$$0\circ x=x\,,\quad x\circ 0=x\,,\quad x\circ x=0\,.$$

In this case, the quasigroup $(Q;\circ,0)$ is a *left*, *right*, *middle loop* respectively. One can find a more detailed information in [6]. Note, the left parastrophe of a middle loop is a left loop and its right parastrophe is a right loop.

✉ fmsokha@ukr.net

A quasigroup is called a *group isotope* if it is isotopic to a group. If

$$x \circ y = \alpha(x) + a + \beta(y), \qquad \alpha 0 = \beta 0 = 0, \tag{1}$$

where $(Q;+,0)$ is a group, then the tuple $(+,\alpha,\beta,a)$ is called a *canonical decomposition defined by* 0 of the group isotope $(Q;\circ)$.

Theorem 1[7]. *Each element of a group isotope uniquely defines its canonical decomposition.*

Proposition 1. *If a commutative middle loop* $(Q;\circ,0)$ *is a group isotope, then its canonical decomposition defined by* 0 *is*

$$x \circ y = \alpha(x) + \alpha(y)$$

where $(Q;+,0)$ *is a group of exponent two, i.e. it is a totally symmetric group.*

P r o o f . Let $(Q;\circ,0)$ be a commutative middle loop which is a group isotope and let (1) be its canonical decomposition defined by 0. Since the group isotope is commutative, then $\alpha = \beta$. Moreover, $0 = 0 \circ 0 = \alpha(0) + a + \beta(0) = a$. The equality $x \circ x = 0$ means $\alpha(x) + \alpha(x) = 0$. Replacing $\alpha(x)$ with x, we obtain $x + x = 0$ which means that the group $(Q;+,0)$ has exponent two.

A *length* of a subterm is called the quantity of functional symbols in it. An identity is called *quadratic*, if in it each variable appears twice.

Theorem 2. [7] *Let a quasigroup satisfy a quadratic identity in three variables satisfying the conditions: the sets of variables of subterms of the length one has two variables and these sets are pairwise different, the set of variables of a subterm of the length two has three elements. Then the quasigroup is isotopic to a group.*

An algebra ${}^{\sigma}({}^{\tau}f) = {}^{\sigma\tau}f \quad {}^{\iota}f = f$ (in brief, $(Q;f)$ is called a *ternary quasigroup*, if it satisfies the following identities:

$$\begin{aligned} &f({}^{(14)}f(x,y,z),y,z) = x\,, && {}^{(14)}f(f(x,y,z),y,z) = x\,, \\ &f(x,{}^{(24)}f(x,y,z),z) = y\,, && {}^{(24)}f(x,f(x,y,z),z) = y\,, \\ &f(x,y,{}^{(34)}f(x,y,z)) = z\,, && {}^{(34)}f(x,y,f(x,y,z)) = z\,. \end{aligned} \tag{2}$$

The operation f is called *invertible*. A ternary quasigroup is *a universal loop*, if each element is neutral, i.e.,

$$f(y,x,x) = f(x,y,x) = f(x,x,y) = y\,. \tag{3}$$

A ternary totally symmetric universal loop is called an *SQS-skein* which is also called *Steiner Ternar idempotent totally symmetric 3-quasigroup*, or *Steiner 3-quasigroup* [1, 2, 8].

A σ*-parastrophe* of an invertible operation f is called an operation ${}^{\sigma}f$ defined by

$${}^{\sigma}f(x_{1\sigma},x_{2\sigma},x_{3\sigma}) = x_{4\sigma} \;:\Leftrightarrow\; f(x_1,x_2,x_3) = x_4, \quad \sigma \in S_4\,,$$

where S_4 denotes the group of all bijections of the set $\{1,2,3,4\}$. Therefore in general, every invertible operation has 24 parastrophes. Some of them can coincide. If all parastrophes coincide, the quasigroup is called *totally symmetric*. Since parastrophes of a quasigroup satisfy the equalities

$${}^{\sigma}({}^{\tau}f) = {}^{\sigma\tau}f \;\text{ and }\; {}^{\iota}f = f\,, \tag{4}$$

then the symmetric group S_4 defines an action on the set Δ_3 of all ternary invertible operations defined on the same carrier. In particular, the fact implies that the number of different parastrophes of an invertible operation is a factor of 24. More precisely, it is equal to $24/|\mathrm{Ps}(f)|$, where $\mathrm{Ps}(f)$ denotes a stabilizer group of f under this action which is called *parastrophic symmetry group* of the operation f. Consequently, a totally symmetric quasigroup is a quasigroup whose parastrophic symmetry group is S_4. If the parastrophic symmetry group of a ternary quasigroup is trivial, then the quasigroup has 24 different parastrophes and it is called *asymmetric.*

Thus, a quadratic identity of the length three is an identity whose individual variables appear twice in it and it has three functional symbols.

For example, all of the identities

$$ {}^{\sigma}f({}^{\tau}f(x,y,z),x,u) = {}^{\nu}f(y,z,u) \text{ where } \sigma,\tau,\nu \in S_4. $$

are ternary quadratic identities of the length three.

2. The main result.

Theorem 3. *In the variety $\mathfrak{U}$ of all ternary universally loops, each quadratic identity of the length three is equivalent to exactly one of the following identities:*

$$ f(z,x,f(x,y,y)) = f(z,u,u), \tag{5} $$

$$ f(x,u,f(y,u,z)) = f(x,y,z), \tag{6} $$

$$ f(f(x,y,z),z,u) = f(y,x,u), \tag{7} $$

$$ f(x,y,f(y,z,u)) = f(x,u,z), \tag{8} $$

$$ f(f(x,y,z),u,y) = f(z,u,x). \tag{9} $$

The identity (6) is equivalent to $f(z,x,f(x,y,y)) = f(z,u,u)$ which is obtained in [2].

Theorem 4. *In the variety of all universal loops $\mathfrak{U}$, the identities (5)–(9) define the following subvarieties:*

1. (5) *defines the variety $\mathfrak{U}$;*

2. (6) *defines the variety $\mathfrak{B}$ of boolean skeins, i.e., the class of all ternary quasigroups $(Q;f)$ such that $f(x,y,z) = x+y+z$ for some boolean group $(Q;+)$;*

3. (7) *the variety* A *of all ternary quasigroups $(Q;f)$ such that $f(x,y,z) = (x \circ y) \overset{\ell}{\circ} z$ for some commutative middle loop $(Q;\circ)$;*

4. (8) *defines the variety ${}^{(13)}\mathrm{A}$ of all ternary quasigroups $(Q;f)$ such that $f(x,y,z) = (z \circ y) \overset{\ell}{\circ} x$ for some commutative middle loop $(Q;\circ)$;*

5. (9) *defines the variety ${}^{(23)}\mathrm{A}$ of all ternary quasigroups $\varepsilon' = \omega'$ such that $f(x,y,z) = (x \circ z) \overset{\ell}{\circ} y$ for some commutative middle loop $(Q;\circ)$.*

The obtained varieties are solutions of A. Krapež's functional equation $F(x,x,y) = y$ and thus it is a partial solution of his problem.

3. Functional equations.

Let Δ_3 be the set of all invertible ternary functions defined on a carrier Q. The relationships (2) and (4) are true for all functions from Δ_3. In other words, the following *hyperidentities* are true over the set Δ_3:

$$ {}^{\sigma}({}^{\tau}F) = {}^{\sigma\tau}F, \ {}^{\iota}F = F, \qquad {}^{(14)}F(F(x,y,z),y,z) = x; $$

$$ {}^{(24)}F(x,F(x,y,z),z) = y; \qquad {}^{(34)}F(x,y,F(x,y,z)) = z; \tag{10} $$

$$ F(x_1,x_2,x_3) = {}^{\sigma}F(x_{1\sigma},x_{2\sigma},x_{3\sigma}), \qquad \sigma \in S_3. $$

The hyperidentities are called *primary*.

Two quasigroup functional equations are called: *equivalent over a set* Q if they have the same solution set over the carrier; *equivalent* if they are equivalent over each set.

Definition 1. [5] Two functional equations are called *parastrophically primarily equivalent* if one can be obtained from the other in a finite number of the following steps: 1) replacing the equation sides; 2) renaming the functional variables; 3) renaming the individual variables; 4) applying the hyperidentities (10).

Lemma 1. *Let* $\varepsilon = \omega$ *and* $\varepsilon' = \omega'$ *be generalized ternary functional equations of the length three. If they are parastrophically primarily equivalent, then there exists a bijection* τ *in* S_3 *and bijections* $\sigma_1, \sigma_2, \sigma_3$ *in* S_4 *such that for an arbitrary solution* (f_1, f_2, f_3) *of* $\varepsilon = \omega$ *the sequence*

$$({}^{\sigma_1} f_{1\tau},\ {}^{\sigma_2} f_{2\tau},\ {}^{\sigma_3} f_{3\tau}) \tag{11}$$

is a solution of the equation $\varepsilon' = \omega'$.

In this case, $(\tau, \sigma_1, \sigma_2, \sigma_3)$ is called *a defining bijection system* of the equations $\varepsilon = \omega$ and $\varepsilon' = \omega'$.

Theorem 5. [3] *Every generalized quadratic ternary quasigroup functional equation of the length three is parastrophically primarily equivalent to exactly one of the following equations*:

$$F_1(z, x, F_2(x, y, y)) = F_3(z, u, u), \tag{12}$$

$$F_1(F_2(x, y, y), z, z) = F_3(x, u, u), \tag{13}$$

$$F_1(F_2(x, y, z), u, u) = F_3(x, y, z), \tag{14}$$

$$F_1(F_2(x, y, z), x, u) = F_3(y, z, u). \tag{15}$$

A functional equation is an identity if all functional variables are parastrophic. Each quasigroup identity defines a variety of quasigroups. Considering the identity as a functional equation means that the solution of the identity is its variety. Two identities are called equivalent, if they define the same variety. Two identities are parastrophically equivalent, if they define parastrophic varieties. Therefore, if identities are parastrophically primarily equivalent, then they are parastrophically equivalent.

Theorem 6. [3] *A triplet* (f_1, f_2, f_3) *of ternary invertible operations defined on set* Q *is a solution of the functional equation (15) if and only if there exist binary invertible operations* $\circ$, $*$, $\Diamond$ *on* Q *such that*

$$\begin{aligned} &f_1(y, x, u) = (x \Diamond y) * u, && \text{(a)} \\ &f_2(x, y, z) = x \overset{r}{\Diamond} (y \circ z), && \text{(b)} \\ &f_3(y, z, u) = (y \circ z) * u. && \text{(c)} \end{aligned} \tag{16}$$

4. Functional equations on universal loops.

A ternary quasigroup is universal if only if its operation is universally neutral. It is easy to see that each triplet of universally neutral operations is a solution of each of the functional equations (12), (13), (14). The solution of the equation (15) is given in the following theorem.

Theorem 7. *A triplet* (f_1, f_2, f_3) *of ternary universally neutral invertible operations defined on set* Q *is a solution of the functional equation (15) if and only if* $f_1 = f_3$ *and there exists a binary commutative middle loop* $(Q; \circ)$ *such that*

$$f_1(x,y,u) = (x \circ y) \overset{\ell}{\circ} u\,, \qquad \text{(a)}$$

$$f_2(x,y,z) = x \overset{r}{\circ} (y \circ z)\,. \qquad \text{(b)} \qquad (17)$$

P r o o f. Let a triplet (f_1, f_2, f_3) of ternary universally neutral invertible operations defined on set Q be a solution of the functional equation (15). According to Theorem 6 there exist binary invertible operations $\circ, *, \Diamond$ on Q such that (16) hold.

Let $x := u$ in (16, a); $x := y = u$, $y := z$ in (16, c) and $x := u = z$ in (16, c):

$$y = (x \Diamond y) * x\,, \qquad y = (x \circ y) * x\,, \qquad y = (y \circ x) * x\,.$$

These identities imply the following equalities:

$$(x \Diamond y) * x = (x \circ y) * x\,, \qquad (x \circ y) * x = (y \circ x) * x$$

which mean that the operation $(\circ)$ is commutative and $(\Diamond)$ coincides with $(\circ)$. Moreover, the equality $y = (y \circ x) * x$ implies that the operation $(*)$ equals $(\overset{\ell}{\circ})$. Thus, $f_1 = f_3$ and (17) hold. Put $x := y$ in (17, a):

$$(x \circ x) \overset{\ell}{\circ} u = u\,.$$

Consequently, the term $x \circ x$ is a left neutral element of the operation $(\overset{\ell}{\circ})$ for all x of Q. Since $(Q; \overset{\ell}{\circ})$ is a quasigroup, then $x \circ x$ is a constant, i.e., $x \circ x = e$ for some element e from Q. It means that the quasigroup $(Q; \circ)$ is a commutative middle loop.

Vice versa, let $(Q; \circ, e)$ be a binary commutative middle loop. We have to prove that the operations f_1 and f_2 defined by (17) are universally neutral, i.e., both satisfy the identities (3). Indeed,

$$f_1(x,x,u) = (x \circ x) \overset{\ell}{\circ} u = e \overset{\ell}{\circ} u = u\,,$$

$$f_1(x,u,x) = (x \circ u) \overset{\ell}{\circ} x = (u \circ x) \overset{\ell}{\circ} x = u\,,$$

$$f_1(u,x,x) = (u \circ x) \overset{\ell}{\circ} x = u.$$

$$f_2(x,x,z) = x \overset{r}{\circ} (x \circ z) = z\,,$$

$$f_2(x,z,x) = x \overset{r}{\circ} (z \circ x) = x \overset{r}{\circ} (x \circ z) = z\,,$$

$$f_2(z,x,x) = z \overset{r}{\circ} (x \circ x) = z \overset{r}{\circ} e = z\,.$$

5. Quadratic identities.

A functional equation is called an *identity* in quasigroups, if all its functional variables are pairwise parastrophic. For example,

$$F\left({}^{\sigma}F(x_1,x_2,x_3), x_1, x_4\right) = {}^{\tau}F(x_2,x_3,x_4)\,, \qquad (18)$$

where $\sigma, \tau \in S_4$, is a quadratic identity of the length 3. This identity is said to be true in a quasigroup $(Q; f)$, if the equality

$$f\left({}^{\sigma}f(x_1,x_2,x_3), x_1, x_4\right) = {}^{\tau}f(x_2,x_3,x_4)\,, \qquad (19)$$

is true for all $x_1,\dots,x_6 \in Q$. It is called an identity in the quasigroup $(Q;f)$.

$$\{1,2\} \in \{\{2\sigma,3\sigma\},\{1\sigma,4\sigma\}\} \cap \{\{1\tau,2\tau\},\{3\tau,4\tau\}\}. \tag{20}$$

Theorem 8.

Let the condition (20) be true. Then (19) is the identity in a ternary universal loop $(Q;f)$ if an only if there exists a commutative middle loop $(Q;\circ,0)$ such that

$$f(x,y,z) = (x \circ y) \overset{\ell}{\circ} z; \tag{21}$$

Let the condition (20) be false. Then the identity (19) is true in a ternary universal loop $(Q;f)$ if and only if there exists a group $(Q;+,0)$ of exponent two such that

$$f(x,y,z) = x + y + z.$$

Note, totally symmetric groups are exactly groups of exponent two.

Proof. The identity (19) is true if universal loop $(Q;f)$ means that the triple $(f, {}^\sigma f, {}^\tau f)$ is a solution of the functional equation (15). Since each parastrophe of a universal loop is also a universal loop, then according to Theorem 7, it is equivalent to the existence a binary commutative middle loop $(Q;\circ,0)$ such that

$$f(x_1,x_2,x_3) = (x_1 \circ x_2) \overset{\ell}{\circ} x_3,$$

$${}^\sigma f(x_1,x_2,x_3) = x_1 \overset{r}{\circ} (x_2 \circ x_3),$$

$${}^\tau f(x_1,x_2,x_3) = (x_1 \circ x_2) \overset{\ell}{\circ} x_3.$$

These equalities are equivalent to

$$f(x_1,x_2,x_3) = x_4 \Leftrightarrow (x_1 \circ x_2) \overset{\ell}{\circ} x_3 = x_4, \qquad \text{(a)}$$

$${}^\sigma f(x_1,x_2,x_3) = x_4 \Leftrightarrow x_1 \overset{r}{\circ} (x_2 \circ x_3) = x_4, \qquad \text{(b)} \tag{23}$$

$${}^\tau f(x_1,x_2,x_3) = x_4 \Leftrightarrow (x_1 \circ x_2) \overset{\ell}{\circ} x_3 = x_4. \qquad \text{(c)}$$

Replace x_i with $x_{i\sigma}$ in (23, *b*) and x_i with $x_{i\tau}$ in (23, c) for all $i = 1,2,3,4$:

$$f(x_1,x_2,x_3) = x_4 \Leftrightarrow (x_1 \circ x_2) \overset{\ell}{\circ} x_3 = x_4,$$

$${}^\sigma f(x_{1\sigma},x_{2\sigma},x_{3\sigma}) = x_{4\sigma} \Leftrightarrow x_{1\sigma} \overset{r}{\circ} (x_{2\sigma} \circ x_{3\sigma}) = x_{4\sigma},$$

$${}^\tau f(x_{1\tau},x_{2\tau},x_{3\tau}) = x_{4\tau} \Leftrightarrow (x_{1\tau} \circ x_{2\tau}) \overset{\ell}{\circ} x_{3\tau} = x_{4\tau}.$$

Using the definition of parastrophes of a ternary invertible operation, we obtain

$$f(x_1,x_2,x_3) = x_4 \Leftrightarrow (x_1 \circ x_2) \overset{\ell}{\circ} x_3 = x_4, \qquad \text{(a)}$$

$$f(x_1,x_2,x_3) = x_4 \Leftrightarrow x_{1\sigma} \overset{r}{\circ} (x_{2\sigma} \circ x_{3\sigma}) = x_{4\sigma}, \qquad \text{(b)} \tag{24}$$

$$f(x_1, x_2, x_3) = x_4 \Leftrightarrow (x_{1\tau} \circ x_{2\tau}) \overset{\ell}{\circ} x_{3\tau} = x_{4\tau}. \qquad \text{(c)}$$

Apply (24, a) to (24, b) and (24, c):

$$f(x_1, x_2, x_3) = (x_1 \circ x_2) \overset{\ell}{\circ} x_3,$$

$$(x_1 \circ x_2) \overset{\ell}{\circ} x_3 = x_4 \Leftrightarrow x_{1\sigma} \overset{r}{\circ} (x_{2\sigma} \circ x_{3\sigma}) = x_{4\sigma},$$

$$(x_1 \circ x_2) \overset{\ell}{\circ} x_3 = x_4 \Leftrightarrow (x_{1\tau} \circ x_{2\tau}) \overset{\ell}{\circ} x_{3\tau} = x_{4\tau}.$$

By definition of ℓ- and r-parastrophes of binary invertible operations, the following relationships are true:

$$f(x_1, x_2, x_3) = (x_1 \circ x_2) \overset{\ell}{\circ} x_3, \qquad \text{(a)}$$

$$x_1 \circ x_2 = x_3 \circ x_4 \Leftrightarrow x_{1\sigma} \circ x_{4\sigma} = x_{2\sigma} \circ x_{3\sigma}, \qquad \text{(b)} \qquad (25)$$

$$x_1 \circ x_2 = x_3 \circ x_4 \Leftrightarrow x_{1\tau} \circ x_{2\tau} = x_{3\tau} \circ x_{4\tau}. \qquad \text{(c)}$$

Thus, the identity (19) is equivalent to the existence of a commutative middle loop $(Q; \circ, 0)$ such that (25) holds. Consider items 1 and 2 of this theorem.

1. If (19) is true, then (24, b) and (24, c) follow from commutativity of the operation $(\circ)$. Therefore, item 1 has been proved.

2. Let there exist a group $(Q; +, 0)$ of exponent two such that $(\circ) = (+)$ and (22) holds, then (25) can be written as follows

$$f(x_1, x_2, x_3) = x_1 + x_2 + x_3,$$

$$x_1 + x_2 = x_3 + x_4 \Leftrightarrow x_{1\sigma} + x_{4\sigma} = x_{2\sigma} + x_{3\sigma},$$

$$x_1 + x_2 = x_3 + x_4 \Leftrightarrow x_{1\tau} + x_{2\tau} = x_{3\tau} + x_{4\tau}.$$

These relationships are true for all σ, τ because the group $(Q; +, 0)$ is totally symmetric. Consequently, the identity (19) is true in the universal loop $(Q; f)$.

Vice versa, let the identity (18) be true in a universal loop $(Q; f)$. Therefore, the relationships (25) are true for some commutative middle loop $(Q; \circ, 0)$. Substitute $(x_1 \circ x_2) \overset{\ell}{\circ} x_3$ for x_4 in the right parts of the relationships (25, b) and (25, c). Considering on τ and σ, we obtain eight possible identities:

$$\left((x_1 \circ x_2) \overset{\ell}{\circ} x_3\right) \circ x_{4\sigma} = x_{2\sigma} \circ x_{3\sigma}, \qquad x_{1\sigma} \circ \left((x_1 \circ x_2) \overset{\ell}{\circ} x_3\right) = x_{2\sigma} \circ x_{3\sigma},$$

$$x_{1\sigma} \circ x_{4\sigma} = \left((x_1 \circ x_2) \overset{\ell}{\circ} x_3\right) \circ x_{3\sigma}, \qquad x_{1\sigma} \circ x_{4\sigma} = x_{2\sigma} \circ \left((x_1 \circ x_2) \overset{\ell}{\circ} x_3\right),$$

$$\left((x_1 \circ x_2) \overset{\ell}{\circ} x_3\right) \circ x_{2\tau} = x_{3\tau} \circ x_{4\tau}, \qquad x_{1\tau} \circ \left((x_1 \circ x_2) \overset{\ell}{\circ} x_3\right) = x_{3\tau} \circ x_{4\tau},$$

$$x_{1\tau} \circ x_{2\tau} = \left((x_1 \circ x_2) \overset{\ell}{\circ} x_3\right) \circ x_{4\tau}, \qquad x_{1\tau} \circ x_{2\tau} = x_{3\tau} \circ \left((x_1 \circ x_2) \overset{\ell}{\circ} x_3\right).$$

Since (20) is false, then at least one of them satisfies the condition of the Theorem 2. Consequently, the commutative middle loop $(Q;\circ,0)$ is isotopic to a group. By Proposition 1, there exists a group $(Q;\oplus,0)$ of exponent two and a self bijection α of the set Q such that $\alpha 0 = 0$ and $x \circ y = \alpha(x) \oplus \alpha(y)$. Therefore, $x \overset{\ell}{\circ} y = \alpha^{-1}(x \oplus \alpha(y))$ and so

$$f(x,y,z) = (x \circ y) \overset{\ell}{\circ} z = \alpha^{-1}(\alpha(x) \oplus \alpha(y) \oplus \alpha(z)) = x + y + z\,,$$

where $x + y := \alpha^{-1}(\alpha(x) \oplus \alpha(y))$. It is easy to see, that the loop $(Q;+,0)$ is isomorphic to the group $(Q;\oplus,0)$. Therefore, $(Q;+,0)$ is a group of exponent two and (22) holds.

P r o o f s of Theorem 3 and Theorem 4.

Since each parastrophe of a universal loop is also universal loop and each triple of universally neutral functions is a solution of the equations (12), (13), (14), then each identity being parastrophically equivalent to the identity of the form (12), (13), (14) is true in any universal loop. That is why such identities are equivalent and define the subvariety A of all ternary universal loops in A. One of this identities is (5) and so the item 1 of Theorem 4 has been proved. It remains to analyze identities of the form (15).

Let A denote the variety of universal loops $(Q; f)$ defined by (21). Let (20) be false. Then according to the item 2 of Theorem 8 there is only one variety, namely the variety of all boolean skeins. Consequently, all such identities are equivalent and (6) is one of them. Thus, the item 2 of Theorem 4 has also been proved.

Let (20) be true. According to Theorem 5 these identities parastrophically equivalent to the identity (22). Remember that for parastrophic symmetry orbit and parastrophic symmetry group of A the relationships

$$\mathrm{Po}(\mathrm{A}) := \left\{ {}^{\sigma}\mathrm{A} \;\middle|\; \sigma \in S_4 \right\}, \quad \mathrm{Ps}(\mathrm{A}) := \left\{ \sigma \in S_3 \;\middle|\; {}^{\sigma}\mathrm{A} = \mathrm{A} \right\},$$

$$|\mathrm{Po}(\mathrm{A})| \cdot |\mathrm{Ps}(\mathrm{A})| = 24\,. \qquad (26)$$

To determine all varieties being parastrophic to A, we'll prove the following lemma.

Lemma 2. *The parastrophic symmetry group of the variety* A *is a dihedral subgroup of* S_4, *namely*

$$\mathrm{Ps}(\mathrm{A}) = D_4 := \{\iota, (12), (34), (12)(34), (13)(24), (1324), (14)(23), (1423)\}\,.$$

D_4 *is a subgroup of the parastrophic symmetry group* $\mathrm{Ps}(f)$ *of any universally loop* $(Q; f)$ *from* A.

P r o o f . Let ν be any element of the parastrophic symmetry group Ps(A). i.e., ${}^{\nu}\mathrm{A} = \mathrm{A}$. By Theorem 8, for any commutative middle loop $(Q;\circ)$ the ν-parastrophe of the loop $(Q; f)$ defined by (21) belongs to A. Consequently, there exists a commutative middle loop $(Q;*)$ such that

$${}^{\nu}f(x_1, x_2, x_3) = (x_1 * x_2) \overset{\ell}{*} x_3 \qquad (27)$$

for all $x_1, x_2, x_3 \in Q$. In other words,

$$^{\nu}f(x_1,x_2,x_3)=x_4 \Leftrightarrow (x_1 * x_2)\overset{\ell}{*}x_3 = x_4 .$$

Replace x_i with $x_{i\nu}$ for all $i=1,2,3,4$:

$$^{\nu}f(x_{1\nu},x_{2\nu},x_{3\nu})=x_{4\nu} \Leftrightarrow (x_{1\nu} * x_{2\nu})\overset{\ell}{*}x_{3\nu} = x_{4\nu} .$$

Using the definitions of parastrophe of ternary and binary quasigroups, we obtain

$$f(f(x,y,z),u,y)=f(z,u,x).$$

Applying (21), we have

$$(x_1 \circ x_2)\overset{\ell}{\circ}x_3 = x_4 \Leftrightarrow x_{1\nu} * x_{2\nu} = x_{3\nu} * x_{4\nu} . \tag{28}$$

Substitute $(x_1 \circ x_2)\overset{\ell}{\circ}x_3 = x_4$ for x_4 in the right part of the relationship. Since $4 \in \{1\nu, 2\nu, 3\nu, 4\nu\}$, then we consider four cases.

Let $4=1\nu$, then we obtain the identity

$$\left((x_1 \circ x_2)\overset{\ell}{\circ}x_3\right) * x_{2\nu} = x_{3\nu} * x_{4\nu} . \tag{29}$$

If $\{1,2\} \neq \{3\nu,4\nu\}$, then by Theorem 2 the loop $(Q;\circ)$ is isotopic to a group. It contradicts to the assumption. Therefore, $\{1,2\}=\{3\nu,4\nu\}$ and $2\nu=3$ and so there are two possible values for ν:

$$\begin{pmatrix}1&2&3&4\\4&3&1&2\end{pmatrix}=(1423) \quad \text{and} \quad \begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}=(14)(23). \tag{30}$$

In both cases the identity (29) is equivalent to

$$((x \circ y)\overset{\ell}{\circ}z) * z = x * y , \tag{31}$$

i.e.,

$$(x * y)\overset{\ell}{*}z = (x \circ y)\overset{\ell}{\circ}z \ ,$$

which means $^{\nu}f = f$. By Theorem 8, it is true for all universal loop $(Q;f)$ from A, then the bijections (30) belong to $\mathrm{Ps}(\mathrm{A})$ and $\mathrm{Ps}(f)$.

Let $2\nu=4$, then (28) is equivalent to the identity

$$x_{1\nu} * \left((x_1 \circ x_2)\overset{\ell}{\circ}x_3\right) = x_{3\nu} * x_{4\nu} . \tag{32}$$

Theorem 2 implies $\{1,2\}=\{3\nu,4\nu\}$ and $1\nu=3$. Therefore, ν has two values:

$$\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}=(13)(24), \qquad \begin{pmatrix}1&2&3&4\\3&4&2&1\end{pmatrix}=(1324). \tag{33}$$

In both cases, the identity (32) is equivalent to (31) and so the permutations (33) belong to $\mathrm{Ps}(\mathrm{A})$ and $\mathrm{Ps}(f)$.

Let $3\nu=4$, then we obtain

$$x_{1\nu} * x_{2\nu} = \left((x_1 \circ x_2)\overset{\ell}{\circ}x_3\right) * x_{4\nu} . \tag{34}$$

Theorem 2 implies the equality $\{1\nu, 2\nu\} = \{1,2\}$. Consequently, there are two values for ν:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} = (34), \qquad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (12)(34).$$

In both cases (34) imply (31).

Therefore, these self bigections belong to $Ps(A)$.

If $\{4\nu\} = 4$, then the self bigections ι and (12) belong to $Ps(A)$.

Since we considered all possible cases, then $Ps(A) = D_4$.

By Lemma 2, ${}^{\nu}f = f$ for all $\nu \in D_4$, then the equality

$$S_4 = D_4 \cup (13)D_4 \cup (23)D_4 \tag{35}$$

implies that it enough to consider (19) when $\sigma, \tau \in \{\iota, (13), (23)\}$. But (20) is true only if $\sigma = (13)$ and $\tau = \iota$. That is why there is exactly one variety which is defined by quadratic identities of the length three satisfying the condition (20) and one of these identities is

$$f({}^{(13)}f(x_1, x_2, x_3), x_1, x_4) = {}^{\iota}f(x_2, x_3, x_4).$$

We obtain it from (19) putting $\sigma = (13)$ and $\tau = \iota$. By definition of parastrophy of ternary operations

$$f(f(x_3, x_2, x_1), x_1, x_4) = f(x_2, x_3, x_4). \tag{36}$$

Replacing these variables, we get

$$f(f(x, y, z), z, u) = f(y, x, u) \tag{37}$$

Since $Ps(A) = D_4$, then $Po(A)$ consists of $|S_4| / |D_4| = 24/8 = 3$ varieties. The equality (35) implies that

$$Po(A) = \{A,\ {}^{(13)}A,\ {}^{(23)}A\}.$$

To find identities which defined the varieties ${}^{(13)}A$ and ${}^{(23)}A$, it is enough to respectively replace f with ${}^{(13)}f$ and f with ${}^{(23)}f$ in (36):

$${}^{(13)}f({}^{(13)}f(x_3, x_2, x_1), x_1, x_4) = {}^{(13)}f(x_2, x_3, x_4),$$

$${}^{(23)}f({}^{(23)}f(x_3, x_2, x_1), x_1, x_4) = {}^{(23)}f(x_2, x_3, x_4).$$

By definition of parastrophy of ternary quasigroups, these identities are equivalent to

$$f(x_4, x_1, f(x_1, x_2, x_3)) = f(x_4, x_3, x_2),$$

$$f(f(x_3, x_1, x_2), x_4, x_1) = f(x_2, x_4, x_3).$$

Replacing the variables, we obtain

$$f(x, y, f(y, z, u)) = f(x, u, z),$$

$$f(f(x, y, z), u, y) = f(z, u, x).$$

Thus, Theorems 3 and 4 have been proved.

6. Identities of the length one and two.

In [4] it is proved the following theorem.

Theorem 9. *Every generalized ternary quasigroup functional equation of the length one is parastrophically primarily equivalent to exactly one of the following equations*:

$$F(x,x,x)=x, \quad (i) \qquad F(x,y,y)=x. \quad (ii)$$

The identity (i) is not quadratic, therefore all quadratic identities belong to the class (ii) and so they can be written as follow

$$^{\sigma}F(x,y,y)=x, \qquad \sigma\in S_4.$$

It is easy to see the truth of the following corollary.

Corollary 1. Every ternary quasigroup quadratic identity of the length one is parastrophically primarily equivalent to at least one of

$$F(x,y,y)=x, \qquad F(y,x,y)=x, \qquad F(y,y,x)=x.$$

For functional equations of the length two it is proved the following theorem in [4].

Theorem 10. *Every generalized ternary quasigroup functional equation of the length two is parastrophically primarily equivalent to exactly one of the following equations*:

$$F_1(x,x,x)=F_2(x,x,x),$$

$$F_1(x,x,x)=F_2(x,y,y), \qquad F_1(x,x,y)=F_2(x,x,y),$$

$$F_1(x,x,x)=F_2(y,y,y), \qquad F_1(x,x,y)=F_2(x,y,y),$$

$$F_1(x,x,y)=F_2(y,z,z), \tag{38}$$

$$F_1(x,y,z)=F_2(x,y,z). \tag{39}$$

Therefore, the quadratic identities of the length two belong to the class (38) or (39). Cosequently, the following statement is true.

Corollary 2. Every ternary quasigroup quadratic identity of the length two is parastrophically primarily equivalent to at least one of

$$^{\tau}F(x,x,y)=F(y,z,z),$$

$$^{\sigma}F(x,y,z)=F(x,y,z),$$

where $\tau,\sigma\in S_4$.

1. *Ganter B., Werner H.* Co-ordinatizing steiner systems // Ann. Discrete Math. – 1980. – 7. – P. 3-24, https://doi.org/10.1016/S0167-5060(08)70167-5
2. *Guelzow A. J.* Semiboolean SQS-skeins // J. Algebraic Combinatorics. – 1993. – 2. – P. 147–153, https://doi.org/10.1023/A:1022411724747
3. *Sokhatsky F. M., Tarasevych A. B.* Classification of generalized ternary quadratic quasigroup functional equations of the length three // Karpat. Math. Publ. – 2019. – 11, No. 1. – P. 179–192, https://doi.org/10.15330/cmp.11.1.179-192
4. *Sokhatsky F., Krainichuk H., Tarasevych A.* A classification of generalized functional equations on ternary quasigroups // Visn. Donetsk Nats. Univ. Ser A: Pryr. Nauky. – 2017. – 1-2. – P. 97–107, https://doi.org/10.31558/1817-2237.2017.1-2.8
5. *Sokhats'kyi F. M.* On classification of functional equations on quasigroups // Ukr. Math. J. – 2004. – 56, No. 9. – P. 1499–1508. https://doi.org/10.1007/s11253-005-0129-3
6. *Sokhatsky F. M.* Parastrophic symmetry in quasigroup theory // Visn. Donetsk Nats. Univ. Ser A: Pryr. Nauky. – 2016. – 1-2. P. 70–83.
7. *Sokhatsky F. M.* On isotopes of groupes II // Ukr. Math. J. – 1995. – 47, No. 12. – P. 1935–1948, https://doi.org/10.1007/BF01060967
8. *Stojaković Z.* Single identities for Mendelsohn and Steiner 3-quasigroups // Bull. Australian Math. Soc. – 1996. – 53, No. 3. – P. 419–424, https://doi.org/10.1017/S0004972700017172

ПРО ТЕРНАРНІ КВАЗІГРУПОВІ КВАДРАТИЧНІ ТОТОЖНОСТІ МАЛОЇ ДОВЖИНИ

Доведено, що кожна квадратична тотожність довжин один, два, три парастрофно-первинно рівносильна принаймні до однієї із заданих тотожностей. Тотожності довжини три були проаналізовані в класі універсальних луп, тобто квазігруп, в яких кожен елемент нейтральний. Доведено, що існує п'ять нерівносильних тотожностей. Перша тотожність визначає клас усіх універсальних луп, друга – многовид булевих мотків (skeins)*, а інші три тотожності визначають три парастрофні многовиди, операції яких – це безповторні композиції бінарних комутативних середніх луп.*

***Ключові слова**: тернарна квазігрупа, квадратичне рівняння, універсальна лупа, квазігрупа Штейнера, тотожність, середня лупа.*

[1] Vasyl' Stus Donetsk National University, Vinnytsia
[2] Khmelnytskyi National University, Khmelnytsk

Received
02.12.20