https://doi.org/10.15407/apmm2020.18.11-13

УДК 512.64+512.56

V. M. Bondarenko[1]✉, M. V. Stepochkina[2], M. V. Stoika[3]

THE COEFFICIENTS OF TRANSITIVENESS OF THE POSETS OF MM-TYPE BEING THE SMALLEST SUPERCRITICAL POSET OF WIDTH 3

We calculate the coefficients of transitiveness for all posets of MM-type being the smallest supercritical poset of width 3 (i.e. posets, that are minimax equivalent to the poset (2, 2, 3)).

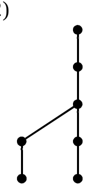
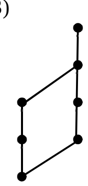
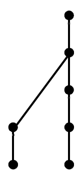
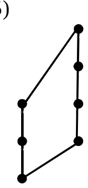
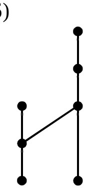
***Ключові слова**: supercritical poset, minimax equivalence, coefficient of transitiveness, MM-type, nodal element, dense subposet.*

Introduction. M. M. Kleiner [9] proved that a poset S is of finite representation type if and only if it does not contain subsets of the form $K_1 = (1;1;1;1)$; $K_2 = (2;2;2)$; $K_3 = (1;3;3)$; $K_4 = (1;2;5)$; and $K_5 = (N;4)$, which are called critical posets; now they are called the Kleiner's posets. On the other hand, Yu. A. Drozd [8] showed that a poset has finite representational type if and only if its Tits quadratic form is weakly positive (i.e., it is positive on the set of non-negative vectors). Consequently, the Kleiner's posets are also critical with respect to weak positiveness of the Tits form, and there are no other such posets. In [2] the first two authors proved that a poset is P-critical (i.e. critical with respect to the positiveness of the Tits form) if and only if it is minimax equivalent to a Kleiner's poset.

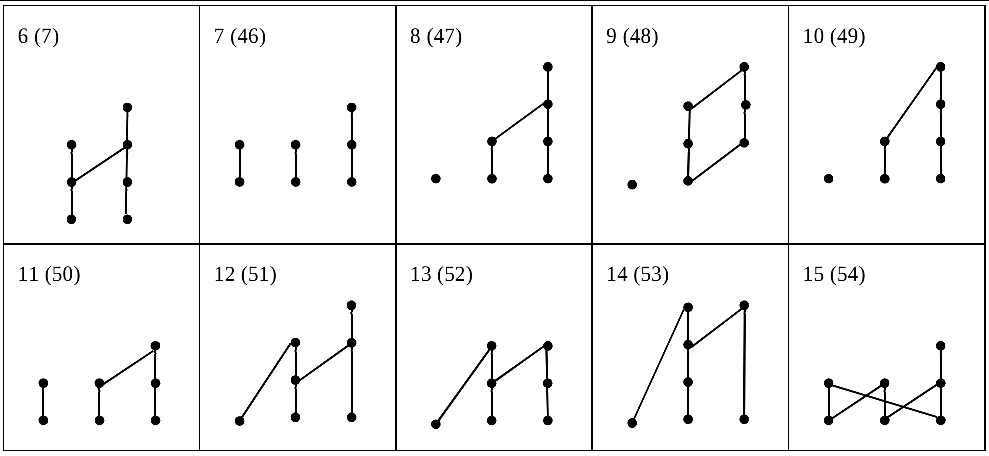
A similar situation takes place for tame posets. L. A. Nazarova [10] proved that a poset S is tame if and only if it does not contain subsets of the form $N_1 = (1;\,1;\,1;\,1;\,1)$, $N_2 = (1;\,1;\,1;\,2)$, $N_3 = (2;2;3)$, $N_4 = (1;\,3;\,4)$, $N_5 = (1;\,2;\,6)$, and $N_6 = (N;\,5)$; these conditions are equivalent to weak non-negativity of the quadratic Tits form. She called these posets supercritical. So the supercritical posets are critical with respect to weak non-negativity of the Tits form and there are no other such posets. The first two authors proved that a poset is critical with respect to non-negativity of the Tits form if and only if it is minimax equivalent to a supercritical poset; all such critical posets were described by them in [3].

In many papers (see e.g. [4–7]) combinatorial properties were studied for various classes of posers. The present paper is devoted to the investigation of combinatorial properties of supercritical posets.

1. The list of posets of MM-type (2; 2; 3). Let P be a fix poset. A poset S is called of MM-type P if S is minimax (in other words, (min, max)-) equivalent to P (the notion of (min, max)-equivalence was introduced in [1]; see also [2]). From the results of [3] it follows that the table below contains all posets (up to isomorphism and duality) of MM-type (2; 2; 3), which is the smallest (in the sense of order) supercritical poset of width 3.

1 (2) 	2 (3) 	3 (4) 	4 (5) 	5 (6) 

✉ vitalij.bond@gmail.com



2. Main result. Let S be a finite poset and $S^2_< := \{(x,y) \mid x,y \in S, x < y\}$. If $(x,y) \in S^2_<$ and there is no z satisfying $x < z < y$, then one says that x and y are *neighboring*. Put $n_w = n_w(S) := \left|S^2_<\right|$ and denote by $n_e = n_e(S)$ the number of pairs of neighboring elements. On the language of the Hasse diagram $H(S)$ (that represents S in the plane), n_e is equal to the number of all its edges and n_w to the number of all its paths, up to parallelity, going bottom-up (two path is called parallel if they start and terminate at the same vertices). The ratio $k_t = k_t(S)$ of the numbers $n_w - n_e$ and n_w we call *the coefficient of transitiveness of* S. If $n_w = 0$ (then $n_e = 0$), we assume $k_t = 0$ (see [5]). Obviously, dual poset have the same coefficient of transitiveness.

The aim of this paper is to calculate k_t for all posets of MM-type $N_3 = (2,2,3) = \{1,2,...,7 \mid 1 \prec 2, 3 \prec 4, 5 \prec 6 \prec 7\}$, which is the smallest super-critical one of width 3.

We write all the coefficients of transitiveness k_t up to the second decimal place.

Theorem. The following holds for posets of MM-type 1–15:

N	n_e	n_w	k_t	N	n_e	n_w	k_t	N	n_e	n_w	k_t
1	6	17	0,65	6	6	13	0,54	11	5	7	0,29
2	7	17	0,59	7	4	5	0,20	12	6	11	0,45
3	6	15	0,60	8	5	11	0,55	13	6	9	0,33
4	7	15	0,53	9	6	11	0,45	14	6	11	0,45
5	6	15	0,60	10	5	9	0,44	15	7	9	0,22

The proof is carried out by direct calculations.

Recall that an element of a poset T is called nodal, if it is comparable with all elements of T. A subposet X of T is said to be dense if there is not $x_1, x_2 \in X$, $y \in T \setminus X$ such that $x_1 < y < x_2$.

Corollary.

a) The poset N_3 *is the only poset of MM-type N3 with the smallest coefficient of transitiveness.*

b) A poset of MM-type N_3 *has the largest coefficient of transitiveness if and only if it contains a dense subposet with three nodal element.*

1. *Bondarenko, V. M.* On (min, max)-equivalence of posets and applications to the Tits forms // Visn. Kyiv Univ., Ser. Fiz. Mat. – 2005. – No. 1. – P. 24–25.
2. *Bondarenko V. M., Styopochkina M. V.* (Min, max)-equivalence of partially ordered sets and the Tits quadratic form // Zb. Pr. Inst. Mat. NAN Ukr. / Problems of Analysis and Algebra / – Kyiv: Inst. of Math. of NAN of Ukraine, 2005. – 2, No. 3. – P. 18–58 (in Russian).
3. *Bondarenko V. M., Stepochkina M. V.* Description of posets critical with respect to the nonnegativity of the quadratic Tits form // Ukrainian Math. J. – 2009. – 61, No. 5. – P. 734–746, https://doi.org/10.1007/s11253-009-0245-6
4. *Bondarenko V. M., Styopochkina M. V.* On properties of the Hasse diagram of nonserial posets with positive quadratic Tits form // Nauk. Visn. Uzhgorod. Univ., Ser. Mat. Inf. – 2016. – 29, No. 2. – P. 31–34.
5. *Bondarenko V. M., Styopochkina M. V.* Coeffcients of transitiveness of *P*-critical posets // Analysis and application, Inst. of Math. of NAN Ukraine. – 2017. – 14, No. 1. – P. 46–51.
6. *Bondarenko V. M., Styopochkina M. V.* Combinatorial properties of *P*-posets of width 2 // Applied problems of mechanics and mathematics. – 2017. – 15 – P. 21–23.
7. *Bondarenko V. M., Styopochkina M. V.* On properties of posets of MM-type (1,3,5) // Nauk. Visn. Uzhgorod. Univ., Ser. Mat. Inf. – 2018. – 32, No. 1. – P. 50–53.
8. *Drozd Yu. A.* Coxeter transformations and representations of partially ordered sets // Funkts. Anal. Prilozhen. – 1974. – 8, No. 3. – P. 34–42 (in Russian).
   English translation: *Funct. Anal. Appl*., 8, No. 3, 219–225 (1974), https://doi.org/10.1007/BF01075695
9. *Kleiner M. M.* Partially ordered sets of finite type // Zap. Nauch. Semin. LOMI. – 1972. – 28. – P. 32–41 (in Russian).
   English translation: *J. Sov. Math*., 3, No. 5, 607–615 (1975), https://doi.org/10.1007/BF01084663
10. *Nazarova L. A.* Partially ordered sets of in-nite type // Izv. Akad. Nauk SSSR Ser. Mat. – 1975. – 39, No. 5. – P. 963–991 (in Russian).

КОЕФІЦІЄНТИ ТРАНЗИТИВНОСТІ Ч. В. МНОЖИН *ММ*-ТИПУ, ЩО Є НАЙМЕНШОЮ СУПЕРКРИТИЧНОЮ Ч. В. МНОЖИНОЮ ШИРИНИ 3

Обчислено коефіцієнти транзитивності для всіх ч. в. множин ММ-типу, що дорівнює найменшій суперкритичній ч. в. множині ширини 3 (тобто ч. в. множин, які є мінімаксно еквівалентними ч. в. множині (2, 2, 3)).

***Ключові слова**: суперкритична частково впорядкована множина, мінімаксна еквівалентність, коефіцієнт транзитивності, ММ-тип, вузловий елемент, щільна частково впорядкована множина.*

[1] Inst. of Math. of NAN of Ukraine, Kyiv
[2] Polissia National Univ., Zhytomyr
[3] Ferenc Rakoczi II. Transcarpathian Hungarian Inst., Beregszasz

Received 01.10.20