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SOME RELATIONSHIPS BETWEEN THE INVARIANT FACTORS OF MATRIX AND ITS SUBMATRIX OVER ELEMENTARY DIVISOR DOMAINS

Relationships between the invariant factors of matrices B and A, where B contains A as its first m rows, are investigated. In particular, we established necessary and sufficient conditions that a matrix A may be augmented with a single row to obtain a matrix B with given invariant factors over elementary divisor domains.

Key words: *Smith normal form, invariant factors, elementary divisor domain.*

An important role in the studying of matrices and their arithmetic properties play the invariant factors and their relationships [1, 3, 6, 7, 9]. In particular, at augmented one matrix with a single row to obtain another matrix are used the relationships between the invariant factors of these matrices. In [4], asserted that a unimodular $m \times n$ $(m < n)$ matrix A over a principal ideal domain may always be augmented with a single row to obtain a unimodular $(m+1) \times n$ matrix B. Over the same area, for arbitrary matrix, R. Thompson [8] showed some relationships between the invariant factors of a matrix A and those of a one row prolongation B. D. Carson [2] obtained similar results in terms of a finitely generated module over principal ideal domains. In this paper, some relationships between the invariant factors of matrices over elementary divisor domains are established. Based on obtained relationships, we give necessary and sufficient conditions that a matrix A may be augmented with a single row to obtain a matrix B.

Let R be an elementary divisor domain [5] with $1 \neq 0$, i.e., every $m \times n$ matrix A over R have diagonal reduction. Namely

$$A \sim \mathrm{E} = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_k, 0, \ldots, 0), \ \varepsilon_{i-1} | \varepsilon_i , \ i = 1, \ldots, k,$$

where the matrix E is called the Smith normal form, ε_i are invariant factors of the matrix A. The notation $a|b$ means that the element a divides the element b, and by the symbol $[a, b]$ we denote the least common multiple of the elements a and b.

Consider the $(m+1) \times n$ matrix B over R. Let's assume that $m \geq n$, B contains A as its first m rows. Suppose, that $\mathrm{rang} A = k < n$, $\mathrm{rang} B = k+1$.

Lemma 1. Let R be an elementary divisor domain, an $(m+1) \times n$ matrix $B \sim \Delta = \mathrm{diag}(\delta_1, \ldots, \delta_{k+1}, 0, \ldots, 0)$, $\delta_i | \delta_{i+1}$, $i = 1, \ldots, k$, and let an $m \times n$ matrix $A \sim \mathrm{E} = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_k, 0, \ldots, 0)$ be a submatrix of B, $\varepsilon_i | \varepsilon_{i+1}$, $i = 1, \ldots, k-1$. Then, if the matrix B is obtained by additions to the matrix A of one row, then

$$\delta_1 | \varepsilon_1 | \delta_2 | \varepsilon_2 | \cdots | \varepsilon_k | \delta_{k+1} . \tag{1}$$

P r o o f. Since the matrix B is obtained by additions to the matrix A of one row, then $B = \begin{pmatrix} A \\ \mathbf{b} \end{pmatrix}$. From that R is an elementary divisor domain, the matrices A and B have diagonal reduction. Hence, we have

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$$B = \begin{pmatrix} \varepsilon_1 & & & & & \\ & \varepsilon_2 & & & \mathbf{0} & \\ & & \ddots & & & \\ & & & \varepsilon_k & & \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ b_1 & b_2 & \cdots & b_k & b_{k+1} & \cdots & b_n \end{pmatrix}, \qquad (2)$$

moreover, $B \sim \mathrm{diag}(\delta_1, \delta_2, \dots, \delta_{k+1}, 0, \dots, 0)$. We know that $\delta_i = \dfrac{\varphi_i}{\varphi_{i-1}}$, $i = 1, \dots, k+1$, where φ_i is a greatest common divisor of minors of order i of the matrix B. To prove the lemma it must be shown that $\delta_i = \dfrac{\varphi_i}{\varphi_{i-1}} | \varepsilon_i$ and $\varepsilon_i | \delta_{i+1} = \dfrac{\varphi_{i+1}}{\varphi_i}$, $i = 1, \dots, k+1$. It is easy to check that the greatest common divisor of minors of order i of the matrix B has the form

$$\varphi_i = (\varepsilon_1 \dots \varepsilon_i,\ \varepsilon_1 \dots \varepsilon_{j-1} b_j \varepsilon_{j+1} \dots \varepsilon_i,\ \varepsilon_1 \dots \varepsilon_{i-1}(b_i, \dots, b_n)), \qquad (3)$$

where for every $i = 1, \dots, k$ the index j takes all natural values from 1 to i.

Let's show that $\dfrac{\varphi_i}{\varphi_{i-1}} | \varepsilon_i$, namely $\dfrac{\varepsilon_i \varphi_{i-1}}{\varphi_i} \in R$. Let's consider

$$\frac{\varepsilon_i \varphi_{i-1}}{\varphi_i} = \frac{\varepsilon_i \left((\varepsilon_1 \dots \varepsilon_{i-1}, \varepsilon_1 \dots \varepsilon_{j-1} b_j \varepsilon_{j+1} \dots \varepsilon_{i-1}, \varepsilon_1 \dots \varepsilon_{i-2}(b_{i-1}, \dots, b_n))\right)}{(\varepsilon_1 \dots \varepsilon_i, \varepsilon_1 \dots \varepsilon_{j-1} b_j \varepsilon_{j+1} \dots \varepsilon_i, \varepsilon_1 \dots \varepsilon_{i-1}(b_i, \dots, b_n))} =$$

$$= \frac{\left((\varepsilon_1 \dots \varepsilon_i, \varepsilon_1 \dots \varepsilon_{j-1} b_j \varepsilon_{j+1} \dots \varepsilon_i, \varepsilon_1 \dots \varepsilon_{i-2} \varepsilon_i (b_{i-1}, \dots, b_n))\right)}{(\varepsilon_1 \dots \varepsilon_i, \varepsilon_1 \dots \varepsilon_{j-1} b_j \varepsilon_{j+1} \dots \varepsilon_i, \varepsilon_1 \dots \varepsilon_{i-1}(b_i, \dots, b_n))} \in R.$$

Similarly, we show that $\varepsilon_i | \dfrac{\varphi_{i+1}}{\varphi_i}$, namely $\dfrac{\varphi_{i+1}}{\varepsilon_i \varphi_i} \in R$.

Lemma 2. *Assume that* (1) *holds. Then the matrix (2) with* $b_j = \dfrac{\delta_1 \dots \delta_j}{\varepsilon_1 \dots \varepsilon_{j-1}}$, $j = 1, \dots, k+1$, $b_j = 0$ *for* $j > k+1$ *has invariant factors precisely* δ_j *for* $j = 1, \dots, k+1$.

P r o o f. As shown in Lemma 1, φ_i is a greatest common divisor of minors of order i of the matrix (2) and has the form (3). Given the conditions of Lemma 2 and the conditions (1), we obtain that

$$\varphi_i = \delta_1 \cdots \delta_i, \ i = 1, \dots, k.$$

Hence, it is obvious, that

$$\varphi_{k+1} = \delta_1 \cdots \delta_{k+1}.$$

Since the i-invariant factor of the arbitrary matrix equal $\dfrac{\varphi_i}{\varphi_{i-1}}$, $i = 1, \dots, k$, where $\varphi_0 = 1$, then the invariant factors of the matrix (2) are precisely δ_j for $j = 1, \dots, k+1$.

From Lemmas 1 and 2 we get the following result.

Theorem 1. Let R be an elementary divisor domain, A be a $m \times n$ matrix over R, $A \sim \mathrm{E} = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_k, 0, \ldots, 0)$, $\varepsilon_i | \varepsilon_{i+1}$, $i = 1, \ldots, k-1$, $\mathrm{rang} A = k < n$. And let $\delta_1, \ldots, \delta_{k+1} \in R$ be nonzero elements such that $\delta_i | \delta_{i+1}$, $i = 1, \ldots, k$. Then the matrix A may be augmented with a single row to obtain a matrix $B = \begin{pmatrix} A \\ \mathbf{b} \end{pmatrix}$ such that $B \sim \Delta = \mathrm{diag}(\delta_1, \ldots, \delta_{k+1}, 0, \ldots, 0)$, $\delta_i | \delta_{i+1}$, $i = 1, \ldots, k$, $\mathrm{rang}\, B = k+1$ if and only if

$$\delta_1 | \varepsilon_1 | \delta_2 | \varepsilon_2 | \cdots | \varepsilon_k | \delta_{k+1} .$$

If $\mathrm{rang} A = \mathrm{rang} B = k \le n$, then we put $b_{k+1} = 0$, $\delta_{k+1} = 0$. Then we get the following result.

Theorem 2. Let R be an elementary divisor domain, A be a matrix over R, $A \sim \mathrm{E} = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_k, 0, \ldots, 0)$, $\varepsilon_i | \varepsilon_{i+1}$, $i = 1, \ldots, k-1$. And let $\delta_1, \ldots, \delta_k \in R$ be nonzero elements such that $\delta_i | \delta_{i+1}$, $i = 1, \ldots, k-1$. Then the matrix A may be augmented with a single row to obtain a matrix $B \sim \Delta = \mathrm{diag}(\delta_1, \ldots, \delta_k, 0, \ldots, 0)$, $\delta_i | \delta_{i+1}$, $i = 1, \ldots, k-1$, if and only if

$$\delta_1 | \varepsilon_1 | \delta_2 | \varepsilon_2 | \cdots | \delta_k | \varepsilon_k .$$

Theorem 3. Let R be an elementary divisor domain, A be an $(n-1) \times (n-1)$ matrix over R, $A \sim \mathrm{E} = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_s, 0, \ldots, 0)$, $\varepsilon_i | \varepsilon_{i+1}$, $i = 1, \ldots, s-1$, $\mathrm{rang} A = s \le n-1$. Then an $n \times n$ matrix $C \sim \Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_t, 0, \ldots, 0)$, $\gamma_i | \gamma_{i+1}$, $i = 1, \ldots, t-1$, $\mathrm{rang} C = t \le n$, exists and contains the matrix A as a submatrix if and only if

$$\begin{aligned} &\gamma_1 | \varepsilon_1 | \gamma_3 , \\ &\gamma_2 | \varepsilon_2 | \gamma_4 , \\ &\cdots\cdots\cdots\cdots \\ &\gamma_{s-1} | \varepsilon_{s-1} | \gamma_{s+1} , \\ &\gamma_s | \varepsilon_s . \end{aligned} \tag{4}$$

P r o o f. (*Necessary*) Let the matrix C exists and contains as a submatrix the matrix A. Consider the $n \times (n-1)$ matrix B. Let B is a submatrix of the matrix C and contains A. According to Theorems 1 and 2, we have

$$\gamma_1 | \delta_1 | \gamma_2 | \delta_2 | \cdots | \gamma_{s-1} | \delta_{s-1} | \gamma_s , \tag{5}$$

and

$$\delta_1 | \varepsilon_1 | \delta_2 | \varepsilon_2 | \cdots | \delta_{s-1} | \varepsilon_{s-1} . \tag{6}$$

It is obvious that (4) follows from (5) and (6).

(*Sufficiency*) Suppose that the matrix A with the Smith normal form E and the Smith normal form Γ are given such that (4) holds. Consider such nonzero elements $\delta_1, \cdots, \delta_{s-1} \in R$, that

$$\delta_i = [\gamma_i, \varepsilon_{i-1}], \quad i = 1, \cdots, s-1, \tag{7}$$

where $\varepsilon_0 = 1$.

Obvious, that from (7) we get the following divisibility $\delta_1 | \delta_2 | \cdots | \delta_{s-1}$, $\gamma_i | \delta_i$ for $i = 1, \cdots, s-1$ and $\varepsilon_{i-1} | \delta_i$ for $i = 2, \cdots, s-1$. From (4) we get, that $\varepsilon_{i-1} | \gamma_{i+1}$ and $\gamma_i | \varepsilon_i$. And that means that

$$\delta_i \| [\gamma_i, \gamma_{i+1}] = \gamma_{i+1}, \; i = 1, \cdots, s-1,$$

and

$$\delta_i \| [\varepsilon_i, \varepsilon_{i-1}] = \varepsilon_i .$$

Therefore $\delta_1, \cdots, \delta_{s-1}$ satisfy (5) and (6). According to Theorems 1 and 2, we may prolong the matrix A by one row to obtain the $n \times (n-1)$ matrix B with the invariant factors δ_j, $j = 1, \ldots, s-1$. Then analogously we may prolong the matrix B by one column to obtain the desired matrix C. ◊

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ДЕЯКІ ВЗАЄМОЗВ'ЯЗКИ МІЖ ІНВАРІАНТНИМИ МНОЖНИКАМИ МАТРИЦІ ТА ЇЇ ПІДМАТРИЦІ НАД ОБЛАСТЯМИ ЕЛЕМЕНТАРНИХ ДІЛЬНИКІВ

Досліджено взаємозв'язки між інваріантними множниками матриць B та A, де B містить A як свої перші m рядків. Зокрема, встановлено необхідні та достатні умови доповнення матриці A одним рядком до матриці B із заданими інваріантними множниками над областями елементарних дільників.

Ключові слова: *нормальна форма Сміта, інваріантні множники, область елементарних дільників.*

НЕКОТОРЫЕ ВЗАИМОСВЯЗИ МЕЖДУ ИНВАРИАНТНЫМИ МНОЖИТЕЛЯМИ МАТРИЦЫ И ЕЁ ПОДМАТРИЦЫ НАД ОБЛАСТЯМИ ЭЛЕМЕНТАРНЫХ ДЕЛИТЕЛЕЙ

Исследованы взаимосвязи между инвариантными множителями матриц B и A, где B содержит A как свои первые m строк. В частности, указаны необходимые и достаточные условия дополнения матрицы A одной строкой к матрице B с заданными инвариантными множителями над областями элементарных делителей.

Ключевые слова: *нормальная форма Смита, инвариантные множители, область элементарных делителей.*

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