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SOLVABLE LIE ALGEBRAS OF DERIVATIONS OF POLYNOMIAL RINGS IN THREE VARIABLES

Let $\mathbf{K}$ be an algebraically closed field of characteristic zero, $A = \mathbf{K}[x_1, x_2, x_3]$ the polynomial ring in three variables and $R = \mathbf{K}(x_1, x_2, x_3)$ the field of rational functions. If L is a subalgebra of the Lie algebra $W_3(\mathbf{K})$ of all $\mathbf{K}$-derivations of A, then RL is a Lie algebra over $\mathbf{K}$ and $\dim_R RL$ will be called the rank of L over R. We study solvable subalgebras L of $W_3(\mathbf{K})$ of rank 3 over R. It is proved that L is isomorphic to a subalgebra of the general affine Lie algebra $aff_3(\mathbf{K})$ if L contains an abelian ideal I of rank 3 over R. If L has an ideal I with $rk_R I = 2$, then L is contained in a subalgebra $\bar{L}$ of $\tilde{W}_3(\mathbf{K}) = Der_{\mathbf{K}} R$ such that $\bar{L}$ is an extension of a subalgebra of $aff_2(F)$ by a subalgebra of dimension ≤ 2, where F is the field of constants of I in R.

Introduction. Let $\mathbf{K}$ be an algebraically closed field of characteristic zero, $A = \mathbf{K}[x_1, x_2, x_3]$ the polynomial ring in three variables and $R = \mathbf{K}(x_1, x_2, x_3)$ the field of rational functions. Recall that a $\mathbf{K}$-linear operator $D : A \to A$ is called a $\mathbf{K}$-derivation on A if D satisfies the Leibniz's rule: $D(fg) = D(f)g + fD(g)$ for all $f, g \in A$. The Lie algebra $W_3(\mathbf{K})$ of all $\mathbf{K}$-derivations on A is a very interesting mathematical object closely connected with groups of symmetries of partial differential equations. In case $\mathbf{K}$ is the field of real or comlpex numbers, all finite dimensional subalgebras of $W_1(\mathbf{K})$ and $W_2(\mathbf{K})$ were described in works of S. Lie, P. Olver, N. Kamran. The natural problem of classification of all finite dimensional subalgebras of $W_3(\mathbf{K})$ remains still open. S. Lie [7] began to study such subalgebras, but his classification even of nilpotent subalgebras is incomplete. U. Amaldi [1, 2] continued study of subalgebras of $W_3(\mathbf{K})$ but his classification is unsatisfactory. Note that the problem of classifying even nilpotent finite-dimensional subalgebras of $W_4(\mathbf{K})$ is wild (i.e. it contains a hopeless problem of classifying pairs of square matrices up to simultaneous similarity [3]).

We study finite dimensional solvable subalgebras of rank 3 over R of the Lie algebra $W_3(\mathbf{K})$ (nilpotent subalgebras of $W_3(\mathbf{K})$ were studied in [10]). The main results of the paper: it is proved in Theorem 1 that a solvable finite dimensional subalgebra L of $W_3(\mathbf{K})$ possessing an abelian ideal of rank 3 over R is isomorphic to a subalgebra of the general affine Lie algebra $aff_3(\mathbf{K})$. If L has an abelian ideal I of rank 2 over R, then L can be embedded in a subalgebra $\bar{L}$ of $\tilde{W}_3(\mathbf{K}) = Der_{\mathbf{K}} R$ such that $\bar{L}$ is an extension of a subalgebra of $aff_2(F)$ by a subalgebra of dimension ≤ 2, where F is the field of constants for the ideal I in the field R.

Notations in the paper are standard. The ground field $\mathbf{K}$ is algebraically closed of characteristic zero. If L is a subalgebra of the Lie algebra $W_3(\mathbf{K})$, then $F = F(L)$ is the field on constants of L in $R = \mathbf{K}(x_1, x_2, x_3)$ (we consider any derivation $D \in W_3(\mathbf{K})$ as derivation of R in the natural way:

$D(f/g) = (D(f)g - fD(g))/g^2$). If V is an n-dimensional vector space over $\mathbf{K}$ and $\mathrm{gl}(V)$ the Lie algebra of all linear operators on V we can consider the semidirect product $\mathrm{gl}(V) \ltimes V$, where V is considered as an abelian Lie algebra. The Lie algebra $\mathrm{gl}(V) \ltimes V$ will be called the general affine Lie algebra and denoted by $aff_n(\mathbf{K})$ (in case $\mathbf{K} = \mathbf{R}$ the Lie algebra $aff_n(\mathbf{R})$ corresponds to the general affine Lie group $GA_n(\mathbf{R})$).

Subalgebras with an abelian ideal of rank 3 over R.

The next two lemmas contain standard facts about derivations (see for example, [8]). More information about derivations of polynomial rings can be found in [9].

Lemma 1 *Let $D_1, D_2 \in W_3(\mathbf{K})$ and $a, b \in R$. Then*

$$[aD_1, bD_2] = ab[D_1, D_2] + aD_1(b)D_2 - bD_2(a)D_1.$$

If $[D_1, D_2] = 0$, then $[aD_1, bD_2] = aD_1(b)D_2 - bD_2(a)D_1$.

Lemma 2 *If $L \subseteq W_3(\mathbf{K})$ and $F = F(L)$ the field of constants for L in R, then FL is a Lie algebra over F. If L is abelian, nilpotent or solvable then so is FL.*

Lemma 3 *Let $D_1, \ldots, D_n$ be a basis of the vector space $W_3(\mathbf{K})$ over the field R. Then $\bigcap_{i=1}^{n} Ker D_i = \mathbf{K}$.*

Proof. Suppose that $\bigcap_{i=1}^{n} Ker D_i \neq \mathbf{K}$ and let $f_1 \in \bigcap_{i=1}^{n} Ker D_i$, $f_1 \in R \setminus \mathbf{K}$. Then there exists a transcendence basis $\{f_1, \ldots, f_n\}$ of R over $\mathbf{K}$ and the subfield $\mathbf{K}(f_1, \ldots, f_n)$ is isomorphic to the field $\mathbf{K}(x_1, \ldots, x_n)$. The function f_1 defines the derivation S of the field $\mathbf{K}(f_1, \ldots, f_n)$ and this derivation can be uniquely extended to the derivation S of $\mathbf{K}(x_1, \ldots, x_n)$ (we keep the same notation for the extended derivation). But $S = \sum_{i=1}^{n} s_i D_i$ for some $s_i \in R$ and therefore $S(f_1) = \sum_{i=1}^{n} s_i D_i(f_1) = 0$ by the choice of the element f_1. This is impossible because $S(f_1) = 1$. The obtained contradiction shows that $\bigcap_{i=1}^{n} Ker D_i = \mathbf{K}$

Corollary 1 *If L is an abelian subalgebra of $W_3(\mathbf{K})$ and $rk_R L = n$, then $\dim_{\mathbf{K}} L = n$.*

Proof. Let $D_1, \ldots, D_n$ be a basis of L over R. Then any element $D \in L$ is of the form $D = \sum_{i=1}^{n} s_i D_i$ for some $s_i \in R$. Since $[D_i, D] = 0 = \sum_{j=1}^{n} D_i(s_j) D_j$ we have that $D_i(s_j) = 0,\ i, j = 1, \ldots, n$. By Lemma 3, $s_i \in \mathbf{K}$ and $D_1, \ldots, D_n$ is a basis of L over $\mathbf{K}$. Thus $\dim_{\mathbf{K}} L = n$.

Theorem 1 *Let L be a solvable subalgebra of the Lie algebra $W_3(\mathbf{K})$. If L has an abelian ideal I of rank 3 over R, then L is isomorphic to a solvable subalgebra of the general affine Lie algebra $aff_3(\mathbf{K})$. In particular $3 \le \dim_{\mathbf{K}} L \le 9$.*

Proof. Take any basis D_1, D_2, D_3 of the ideal I over the field R. Then any element $D \in L$ can be written in the form

$$D = s_1 D_1 + s_2 D_2 + s_3 D_3, \quad s_i \in R.$$

Since $[D_i, D] = D_i(s_1)D_1 + D_i(s_2)D_2 + D_i(s_3)D_3 \in I$ we have by Lemma 4 that $D_i(s_j) \in \mathbf{K}$, $i, j = 1,2,3$. So we can correspond to any element $D \in L$ the matrix

$$B_D = \begin{pmatrix} D_1(s_1) & D_1(s_2) & D_1(s_3) \\ D_2(s_1) & D_2(s_2) & D_2(s_3) \\ D_3(s_1) & D_3(s_2) & D_3(s_3) \end{pmatrix} \in M_3(\mathbf{K}) \quad (1)$$

Denote by S the set of all columns of such matrices B_D, where D runs over the subalgebra L. Since $S \subseteq \mathbf{K}^3$, the three-dimension vector space over $\mathbf{K}$, we have $d = rk_{\mathbf{K}} S \le 3$. If $d = 0$, then all columns for all $D \in L$ are zero and therefore $s_i \in \mathbf{K}$, $i = 1,2,3$ by Lemma 3. This means $L = I$. So we can assume that $d \ge 1$.

<u>Case 1.</u> $d = 1$. Then there exists an element $D \in L \setminus I$ which can be written in the form $D = s_1D_1 + s_2D_2 + s_3D_3$ such that all columns of S are proportional to the column $(D_1(s_1), D_2(s_1), D_3(s_1))^T$ (here $\cdot^T$ denotes the transpose of the row) of the corresponding matrix B_D. Take any element $(D_1(t), D_2(t), D_3(t))^T \in S$. Then there exists $\gamma \in \mathbf{K}$ such that

$$(D_1(t), D_2(t), D_3(t))^T = \gamma(D_1(s_1), D_2(s_1), D_3(s_1))^T.$$

It follows from the last equality that

$$D_1(t - \gamma s_1) = D_2(t - \gamma s_1) = D_3(t - \gamma s_1) = 0.$$

By Lemma 3 we obtain $t - \gamma s_1 = \delta$ for some $\delta \in \mathbf{K}$, i.e. $t = \gamma s_1 + \delta$. The latter means that for any element $D \in L$, $D = t_1D_1 + t_2D_2 + t_3D_3$, $t_i \in R$ the corresponding matrix B_D has the columns $(D_1(t_i), D_2(t_i), D_3(t_i))^T$ $i = 1,2,3$ with $t_i = f_i(s)$, $\deg f_i \le 1$, $f_i \in \mathbf{K}[t]$. Since $(D_1(s_1), D_2(s_1), D_3(s_1))^T$ is nonzero we can assume without loss of generality that $D_1(s_1) = 1, D_2(s_1) = \gamma_2, D_3(s_1) = \gamma_3$ for some $\gamma_2, \gamma_3 \in \mathbf{K}$. Put

$$D_{1'} = D_1,\ D_{2'} = D_2 - \gamma_2 D_1,\ D_{3'} = D_3 - \gamma_3 D_1.$$

Then $D_{1'}(s_1) = 1, D_{2'}(s_1) = 0, D_{3'}(s_1) = 0$ and $D_{1'}, D_{2'}, D_{3'}$ form a basis of I over R. Let $D = t_1D_1 + t_2D_2 + t_3D_3$ be an arbitrary element in L and $t_i = \gamma_i s_i + \delta_i$, $i = 1,2,3.$ Then the map $\varphi : L \to aff_3(\mathbf{K})$ which is defined by the rule: $\varphi(D_i) = x_i$, $\varphi(s_1D_i) = x_1x_i$ and further by linearity, is an embedding of L into the Lie algebra $aff_3(\mathbf{K})$.

<u>Case 2.</u> $d = rk_{\mathbf{K}} S = 2$. Then there exist linearly independent columns on the set S of the form

$$(D_1(s_1), D_2(s_1), D_3(s_1))^T, (D_1(s_2), D_2(s_2), D_3(s_2))^T \quad (2)$$

(these columns can belong to different matrices $B_D, D \in L$). Therefore any column $(D_1(t), D_2(t), D_3(t))^T \in S$ is a linear combination of columns in (2). One can easily show that $t = f(s_1, s_2)$ for some polynomial $f \in \mathbf{K}[u, v]$, $\deg f \le 1$. Note that the rank of the matrix

$$\begin{pmatrix} D_1(s_1) & D_1(s_2) \\ D_2(s_1) & D_2(s_2) \\ D_3(s_1) & D_3(s_2) \end{pmatrix} \quad (3)$$

is equal to 2. Without loss of generality one can assume that the first and second rows of this matrix are linearly independent. But then there exist $\gamma_1, \gamma_2 \in \mathbf{K}$ such that

$$(1,0) = \gamma_1 (D_1(s_1), D_1(s_2)) + \gamma_2 (D_2(s_1), D_2(s_2)). \quad (4)$$

Denoting $D_{1'} = \gamma_1 D_1 + \gamma_2 D_2$ we have $D_{1'}(s_1) = 1, D_{1'}(s_2) = 0$. Analogously one can find $\delta_1, \delta_2 \in \mathbf{K}$ such that the element $D_{2'} = \delta_1 D_1 + \delta_2 D_2$ has properties $D_{2'}(s_1) = 0, D_{2'}(s_2) = 1$.

Further, the third row of the matrix (3) is a linear combination of the first and second rows and therefore $(D_3 - \mu_1 D_1 - \mu_2 D_2)(s_i) = 0, i = 1,2$. Denoting $D_{3'} = D_3 - \mu_1 D_1 - \mu_2 D_2$ we obtain $D_{i'}(s_j) = \delta_{ij}$, $i = 1,2,3$, $j = 1,2$. If $D \in L$ is an arbitrary element, then $D = t_1 D_1 + t_2 D_2 + t_3 D_3$ for some $t_1, t_2, t_3 \in R$. Since $t_i = f_i(s_1, s_2)$, $\deg f_i \le 1$ we see that L can be embedded in the Lie algebra $aff_3(\mathbf{K})$.

Case 3 $rk_{\mathbf{K}} S = 3$ can be considered analogously.

Subalgebras with abelian ideals of $rk \le 2$ over R.

Lemma 4 *Let L be a subalgebra of the Lie algebra $W_n(\mathbf{K})$ and I be an ideal of L. If $F = F(I)$ is the field of constants for I in R, then $D(F) \subseteq F$ for any element $D \in L$.*

Proof. Let $D \in L$ and $r \in F$ be arbitrarily chosen. Then for any $D_1 \in I$ we have $D_1(r) = 0$ and therefore

$$0 = D(D_1(r)) = D_1(D(r)) + [D, D_1](r).$$

Since $[D, D_1] \in I$ we have $[D, D_1](r) = 0$ and consequently $D_1(D(r)) = 0$. The latter means that $D(r) \in F$ because the element D_1 was arbitrarily chosen in the ideal I. Thus $D(F) \subseteq F$.

Theorem 2 *Let L be a solvable finite dimensional subalgebra of the Lie algebra $W_3(\mathbf{K})$ with $\mathrm{r}k_R L = 3$. If L has an ideal I of rank 2 over R and $F = F(L)$ is the field of constants of I in R, then the Lie algebra L is contained in the subalgebra $\overline{L} = F\overline{I} + L$ of $W_3(\mathbf{K})$ where $\overline{I} = (RI) \cap L$. The Lie algebra $\overline{L}$ is solvable, $F\overline{I}$ is its ideal of rank 2 over R which is isomorphic to a subalgebra of $aff_2(F)$. The Lie algebra $\overline{L}$ is an extension of the ideal $F\overline{I}$ by a Lie algebra of dimension 1 or 2 over $\mathbf{K}$.*

Proof. The intersection $\overline{I} = (RI) \cap L$ is an ideal of the Lie algebra L with $\mathrm{r}k_R \overline{L} = 2$ and $\dim_{\mathbf{K}} L / \overline{I} \le 2$ (see [8]). Let F be the field of constants for I in R. Since $D(F) \subseteq F$ for any $D \in L$ (by Lemma 4), the subalgebra $F\overline{I}$ of the algebra $W_3(\mathbf{K})$ is an ideal of the Lie algebra $F\overline{I} + L$. One can easily show that $\mathrm{r}k_{\mathbf{R}} \overline{I} = 2$. By Theorem 1 of the paper [6], the Lie algebra $F\overline{I}$ (as a Lie algebra over the field F) is isomorphic to a subalgebra of the Lie algebra $aff_2(F)$. Since $\dim_{\mathbf{K}} L / \overline{I} \le 2$, it holds obviously $\dim_{\mathbf{K}} L + F\overline{I} / F\overline{I} \le 2$. Note that the Lie algebra $L + F\overline{I}$ is in general case of infinite dimension over $\mathbf{K}$ although $\dim_F F\overline{I} \le 7$ (the sum $F\overline{I} + L$ is not in general a Lie algebra over F but only over the field $\mathbf{K}$).The proof is complete.

Further notations are taken from Theorem 2. Let $I_1 = \mathbf{K} D_1$ be a one-dimensional ideal of L lying in I and $\mathbf{K} D_2 + I_1$ be an ideal of the quotient algebra L/I_1 lying in I/I_1 (such ideals do exist because L is solvable and $\mathbf{K}$ is algebraically closed). Let $\mathbf{K}D_3+\bar{I}$ be one-dimensional ideal of the Lie algebra $L/\bar{I}$. Then D_1,D_2,D_3 are linearly independent over R and form a basis of RL over R. By the choice of D_1 and D_2 there exist $\lambda_1,\lambda_2\in K$ and $g_2\in F$ such that

$$[D_3,D_1]=\lambda_1 D_1,\ [D_3,D_2]=\lambda_2 D_2+g_2D_1.$$

The next statement gives more detailed description of the Lie algebra $\overline{L}=F\bar{I}+L$.

Proposition 1 *Let $L\subseteq W_3(\mathbf{K})$ be a solvable finite dimensional subalgebra of rank* 3 *over* R *with* $\dim L>6$. *Under conditions of Theorem 2 either there exist $r_1,r_2\in R$ with $D_i(r_j)=\delta_{ij}, i,j=1,2$ and every element $D\in F\bar{I}$ is of the form $D=f_1(r_1,r_2)D_1+f_2(r_1,r_2)D_2$, $f_i\in\mathbf{K}[t_1,t_2]$, $\deg f_i\le 1$ or there exists $r_i\in R$, $i=1$ or $i=2$ with $D_i(r_j)=\delta_{ij}$ and every element $D\in F\bar{I}$ is of the form $D=g_1(r_i)D_1+g_2(r_i)D_2$, $\deg g_j\le 1$. Then $D_3(r_1)=-\lambda_1r_1-g_2r_2$, $D_3(r_2)=-\lambda_2r_2$. If $\dim_{\mathbf{K}}L/\bar{I}=2$, then there exists $\overline{D}\in L\setminus(\mathbf{K}D_3+\bar{I})$ such that $\overline{D}=r_3D_3+s_2D_2$, $r_3\in R$, $D_3(r_3)=1$, $D_1(r_3)=D_2(r_3)=0$, $D_1(s_2)=0$, and in this case $\lambda_1=0, g_2=0$, $s_2=\lambda_2r_2r_3+f$, $f\in\mathbf{K}$.*

Proof. Repeating considerations from the proof of Theorem 1 one can find either elements r_1,r_2 with $D_i(r_j)=\delta_{ij}, i,j=1,2$ or an element $r\in R$ such that either $D_1(r)=1, D_2(r)=\gamma$ or $D_1(r)=\delta, D_2(r)=1$ using only transformations of columns of the matrix $B_D=\begin{pmatrix}D_1(s_1) & D_1(s_2)\\ D_2(s_1) & D_2(s_2)\end{pmatrix}$. If $\delta\ne 0$ we can consider elements $D_{2'}=D_2-\delta D_1, D_{1'}=D_1$, and in this case $D_{1'}(r)=0, D_{2'}(r)=1$. So we can assume that either $D_1(r)=1, D_2(r)=0$ or $D_1(r)=0, D_2(r)=1$ and r is either r_1 or r_2.

Let us consider the action of elements D_i on r_i,s_j, $i=1,2,3, j=2,3$.

Since $D_1(r_1)=1$ we have $D_3(D_1(r_1))=0$ and therefore

$$D_1(D_3(r_1))=D_3(D_1(r_1))-[D_3,D_1](r_1)=0-\lambda_1D_1(r_1)=-\lambda_1.$$

It follows from the equalities $D_1(D_3(r_1))=-\lambda_1$ and $D_1(-\lambda_1r_1)=-\lambda_1$ that $D_1(D_3(r_1)+\lambda_1r_1)=0$, i.e. $D_3(r_1)=-\lambda_1r_1+s'$ for some $s'\in KerD_1$. Analogously the equality

$$D_2(d_3(r_1))=D_3(D_2(r_1))-[D_3,D_2](r_1)$$

implies $D_3(r_1)=-g_2r_2+s''$ for some $s''\in KerD_2$. Applying D_1 to both sides of the obtained equality $-\lambda_1r_1+s'=-g_2r_2+s''$ we get $-\lambda_1=D_1(s'')$. After applying D_2 to the same equality we get $D_2(s')=-g_2$. But then $s''+\lambda_1r_1\in KerD_1$. Since $s''+\lambda_1r_1\in KerD_2$ we have $s''+\lambda_1r_1\in KerD_1\cap KerD_2=F$. Thus $s''=-\lambda_1r_1+v_1$ for some $v_1\in F$ It follows from the equality $-\lambda_1r_1+s'=-g_2-\lambda_1r_1+v_1$ that $s'=-g_2r_2+v_1$. Finally we get

$$D_3(r_1)=-\lambda_1r_1-g_2r_2+v_1,\ v_1\in F.$$

Analogously it follows from the equalities

$$D_2(D_3(r_2))=D_3(D_2(r_2))-[D_3,D_2](r_2)=0-(\lambda_2D_2+g_2D_1)(r_2)=-\lambda_2$$

that $D_3(r_2)=-\lambda_2r_2+t'$ for some $t'\in KerD_2$ and finally

$$D_3(r_2) = -\lambda_2 r_2 + v_2, v_2 \in F.$$

Without loss of generality we can change D_3 by $D_{3'} = D_3 - v_1 D_1 - v_2 D_2$. Then $D_{3'}(r_1) = -\lambda_1 r_1 - g_2 r_2, D_{3'}(r_2) = -\lambda_2 r_2$. Returning to the old notation we have $D_3(r_1) = -\lambda_1 r_1 - g_2 r_2, D_3(r_2) = -\lambda_2 r_2$.

Let now $\dim_{\mathbf{K}} L / \bar{I} = 2$ and $\bar{D} = r_3 D_3 + s_1 D_1 + s_2 D_2$ be any element of $L \setminus (\mathbf{K} D_3 + I)$. Then

$$[\bar{D}, D_3] = [r_3 D_3 + s_1 D_1 + s_2 D_2, D_3] =$$
$$= -D_3(r_3) D_3 - D_3(s_1) D_1 - s_1 [D_1, D_3] - D_3(s_2) D_2 - s_2 [D_2, D_3] =$$
$$= -D_3(r_3) D_3 + (-D_3(s_1) + \lambda_1 s_1 + s_2 g_2) D_1 + (-D_3(s_2) + \lambda_2 s_2) D_2.$$

It follows from these equalities that $D_3(r_3) = -\gamma,$ where γ is taken from the equality $[\bar{D}, D_3] = \gamma D_3 + \mathring{D}$, where $\mathring{D} \in \bar{I}$. Analogously the equality

$$[r_3 D_3 + s_1 D_1 + s_2 D_2, D_1] = \mu D_1$$

for some $\mu \in \mathbf{K}$ implies $D_1(r_3) = 0, D_1(s_2) = 0$. The equality

$$[r_3 D_3 + s_1 D_1 + s_2 D_2, D_2] = f_1 D_1 + f_2 D_2$$

for some $f_1, f_2 \in F$ yields $D_3(r_3) = 0$. Summarizing we get

$$D_1(r_3) = D_2(r_3) = 0, \qquad D_3(r_3) = 1, \qquad D_1(s_2) = 0. \tag{5}$$

Since $[\bar{D}, D_1] = \theta D_1$ for some $\theta \in \mathbf{K}$ we have

$$[r_3 D_3 + s_1 D_1 + s_2 D_2, D_3] = (\lambda_1 r_3 - D_1(s_1)) D_1$$

and therefore $\lambda_1 r_3 - D_1(s_1) = \theta$. Thus $D_1(s_1) = \lambda_1 r_3 + \theta, \theta \in \mathbf{K}$. Further $[\bar{D}, D_2] = f_1 D_1 + f_2 D_2$ for some $f_1, f_2 \in F$. Analogously $[r_3 D_3 + s_1 D_1 + s_2 D_2, D_2] = (r_3 g_2 - D_2(s_1)) D_1 + (\lambda_2 r_2 - D_2(s_2)) D_2$ and therefore

$$D_2(s_1) = g_2 r_3 - f_2, \qquad D_2(s_2) = \lambda_2 r_3 - f_2. \tag{6}$$

But we have

$$s_1 = g_2 r_2 r_3 - r_2 f_2 + f_3, s_2 = \lambda_2 r_2 r_3 - r_2 f_2 + f_4$$

for some $f_3, f_4 \in F$. It was proved early that $D_1(s_1) = \lambda_1 r_3 + \theta, \theta \in \mathbf{K}$, so we have $s_1 = \lambda_1 r_1 r_3 + \theta r_1 + f_5$ for some $f_5 \in F$. Applying D_2 to the both sides of the equality

$$\lambda_1 r_1 r_3 + \theta r_1 + f_5 = g_2 r_2 r_3 - r_2 f_2 + f_3 \tag{7}$$

we get $g_2 r_3 - f_2 = 0$. But r_1, r_2, r_3 are linearly independent over F, so the last equality yields $g_2 = 0$. The equality (7) is now of the form

$$\lambda_1 r_1 r_3 + \theta r_1 + f_5 = -r_2 f_2 + f_3.$$

Applying D_2 to the both sides of this equality we get $f_2 = 0$. Therefore $\lambda_1 r_1 r_3 + \theta r_1 + f_5 = f_3$. Applying D_1 to the both sides of the last equality we get $\lambda_1 r_3 + \theta = 0$. Since $r_3 \not\in \mathbf{K}$ we have $\lambda_1 = 0$ and therefore $s_1 = 0.$ Analogously we can assume that $f_4 = 0$ and $s_2 = \lambda_2 r_2 r_3.$ So we have

$$s_1 = 0, s_2 = \lambda_2 r_2 r_3, g_2 = 0, f_2 = 0, \lambda_1 = 0.$$

These equalities means that

$$[D_3, D_1] = 0, [D_3, D_2] = \lambda_2 D_2, \bar{D} = r_3 D_3 + s_2 D_2,$$

where $s_2 = \lambda_2 r_2 r_3, D_i(r_j) = \delta_{ij}$, $i, j = 1,2,3$. The proof is complete.

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РОЗВ'ЯЗНІ АЛГЕБРИ ЛІ ДИФЕРЕНЦІЮВАНЬ КІЛЕЦЬ МНОГОЧЛЕНІВ ВІД ТРЬОХ ЗМІННИХ

Нехай K *– алгебраїчно замкнене поле характеристики нуль,* $A=\mathbf{K}[x_1,x_2,x_3]$ *– кільце многочленів від трьох змінних і* $R=\mathbf{K}(x_1,x_2,x_3)$ *– поле раціональних функцій. Якщо* L *– підалгебра алгебри Лі* $W_3(\mathbf{K})$ *всіх* $\mathbf{K}$*-диференціювань кільця* A*, то* RL *є алгеброю Лі над* $\mathbf{K}$ *і* $\dim_R RL$ *називається рангом алгебри* L *над* R*. Вивчаються підалгебри* L *рангу 3 над* R *алгебри Лі* $W_3(\mathbf{K})$*. Доведено, що якщо* L *містить абелевий ідеал* I *рангу 3 над* R*, то* L *ізоморфна підалгебрі загальної афінної алгебри Лі* $aff_3(\mathbf{K})$*. Якщо* L *має ідеал* I *з* $rk_R I=2$*, то* L *міститься в підалгебрі* $\bar{L}$ *алгебри* $\widetilde{W}_3(\mathbf{K})=Der_{\mathbf{K}}R$*, де* $\bar{L}$ *– розширення деякої підалгебри із* $aff_2(F)$ *за допомогою підалгебри розмірності* ≤ 2*, а* F *– поле констант для* I *в* R*.*

РАЗРЕШИМЫЕ АЛГЕБРЫ ЛИ ДИФФЕРЕНЦИРОВАНИЙ КОЛЕЦ МНОГОЧЛЕНОВ ОТ ТРЕХ ПЕРЕМЕННЫХ

Пусть K *– алгебраически замкнутое поле характеристики нуль,* $A=\mathbf{K}[x_1,x_2,x_3]$ *– кольцо многочленов от трех переменных и* $R=\mathbf{K}(x_1,x_2,x_3)$ *– поле рациональных функций. Если* L *-подалгебра алгебры Ли* $W_3(\mathbf{K})$ *всех* $\mathbf{K}$*-дифференцирований кольца* A*, то* RL *является алгеброй Ли над* $\mathbf{K}$ *и* $dim_R RL$ *называется рангом алгебры* L *над* R*. Исследуются підалгебры* L *ранга 3 над* R *алгебры Ли* $W_3(\mathbf{K})$*. Доказано, что если* L *содержит абелев идеал* I *ранга 3 над* R*, то* L *изоморфна подалгебре общей афинной алгебры Ли* $aff_3(\mathbf{K})$*. Если* L *содержит идеал* I *с* $rk_R I=2$*, то* L *содержится в подалгебре* $\bar{L}$ *алгебры* $\widetilde{W}_3(\mathbf{K})=Der_{\mathbf{K}}R$*, где* $\bar{L}$ *– расширение некоторой подалгебры из* $aff_2(F)$ *с помощью подалгебры размерности* ≤ 2*, а* F *– поле констант для* I *в* R*.*

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