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ADEQUATE PROPERTIES OF THE ELEMENTS WITH ALMOST STABLE RANGE 1 OF A COMMUTATIVE ELEMENTARY DIVISOR DOMAIN

(In boring memory of V. I. Andriychuk on the 70 th anniversary of his birth)

It is shown that in a commutative elementary divisor domain which is not a ring of stable range 1 exist nonzero and nonunit elements with almost stable range 1.

The problem of diagonalization of matrices is a classic one. Specific role in modern research on elementary divisor rings is played by a K – theoretical invariant as the stable range [4]. Important role in studying of the elementary divisor rings played a Hermite rings, i.e. ring in which 1×2 and 2×1 matrices over this ring have diagonal reduction. Note that any Hermite ring is a Bezout ring i.e. a ring in which any finitely generated ideal is principal. We have the following result.

Theorem 1[4]. A commutative Bezout ring is Hermite ring if and only if it is a ring of stable range 2.

Recall, that a ring R is a ring of stable range 2 if for any elements $a,b,c\in R$ the equality $aR+bR+cR=R$ implies that there are some elements λ,μ such that

$$(a+c\lambda)R+(b+c\mu)R=R.$$

Recall, that a ring R is a ring of stable range 1 if for any elements $a,b\in R$ the equality $aR+bR=R$ implies that there are some element t such that $(at+b)R=R$.

Let R – commutative elementary divisor domain which is not a ring of stable range 1.
By [2] there exists nonzero and nonunit element $a\in R$ with almost stable range 1 (i.e. for any elements $b,c\in R$ such that $aR+bR+cR=R$ exists element t that $aR+(bt+c)R=R$). In this paper we describe algebraic properties these element $t\in R$.

By [2] we have that the problem "is every commutative Bezout domain an elementary divisor ring" is equivalent to the problem does every commutative Bezout domain contain a nonunit element with almost stable range 1. In this article gives a more precise description of this elements.

All rings considered will be commutative and have identity. Element $a\in R$ of a commutative ring is called a neat element if for any elements $b,c\in R$ such that $bR+cR+aR=R$ we have $a=rs$ where $rR+bR=R$, $sR+cR=R$, $rR+sR=R$.

Theorem 2[3]. Let R be a commutative Bezout domain. An element a is a neat element if the factor-ring R/aR is a clean ring.

Recall that a ring is called clean if each element is the sum of the unit and an idempotent.

A commutative ring R is said to be a ring of neat range 1 if for any $a,b\in R$ such that $aR+bR=R$ there exists $t\in R$ such that $a+bt\in R$ is a neat element. We have a next result.

Theorem 3[3]. A commutative Bezout domain is an elementary divisor ring if it is a ring of neat range 1.

Recall that a commutative ring R is called an elementary divisor ring if every matrix A over R admits diagonal reduction, that is there exist invertible matrices P and Q such that PAQ is a diagonal matrix, (d_i) for which d_i is a divisor of d_{i+1}.

Recall that a ring R is an exchange ring if for any element $a \in R$ there exists an idempotent e such that $e \in aR$ and $1-e \in (1-a)R$ [13].

A ring R is a ring of idempotent stable range 1 if the condition $aR+bR=R$ for all elements $a,b \in R$ implies that there exists an idempotent $e \in R$ such that $a+be$ is an invertible element of the ring R.

We have the following result.

Theorem 4[1]. Let R be a commutative ring. The following properties are equivalent:

1) R is an exchange ring,
2) R is a clean ring,
3) R is a ring of idempotent stable range 1.

Proposition. Let R be a commutative ring. Nonzero element $a \in R$ is element with almost stable range 1 if and only if a factor-ring R/aR is a ring of stable range 1.

P r o o f. Denote $\bar{R} = R/aR$ and $\bar{b} = b + aR$, $\bar{c} = c + aR$. If $\bar{R}$ is a ring of stable range 1 and $\bar{b}\bar{R} + \bar{c}\bar{R} = \bar{R}$, then exists element $\bar{t} \in R$ such that $(\bar{b}\bar{t} + \bar{c})\bar{R} = \bar{R}$. Since $\bar{R} = R/aR$ and by [1] we have $aR + (bt+c)R = R$, where $\bar{t} = t + aR$, i.e. a is element with almost stable range 1. We notice, that condition $aR + (bt+c)R = R$ implies $\bar{b}\bar{R} + \bar{c}\bar{R} = \bar{R}$. Proposition is proved.

Nonzero element a of a commutative ring R is said to be adequate to the element $b \in R$ (aAb denote this fact) if we can find such elements $r, s \in R$ that the decomposition $a = rs$ satisfying the following properties:

1) $rR + bR = R$,
2) $s'R + bR \neq R$ for any noninvertible divisor s' of element s.

If for any element $b \in R$ we have aAb then we say that element a is adequate. If any nonzero element of a ring R is an adequate element then R is called an adequate ring. An addition we notice simple fact: for any nonzero element a of R we have aAa. The most obvious examples of adequate elements are units, square free elements and factorial elements [4]. By [2] we have that an adequate element is a neat element.

The main result of this paper is a next Theorem.

Theorem 5. Let R be a commutative elementary divisor domain, which is not a ring of stable range 1. Then there exists nonunit and nonzero element $a \in R$ and for any $b, c \in R$ such that $aR + bR + cR = R$ there exists element $t \in R$ such that $aR + (bt+c)R = R$ and aAt.

P r o o f. Let R be a commutative elementary divisor domain. By [4] R is a Bezout domain. By Theorem 3 R is a ring of a neat range 1. Since R is not a ring of stable range 1, then in R exists nonzero and nonunit neat element a. By Theorem 2 we have that $\bar{R} = R/aR$ is a clean ring. By Theorem 3, we have that $\bar{R} = R/aR$ is a ring idempotent stable range 1. Let $\bar{b} = b + aR$, $\bar{c} = c + aR$. Then we have if $\bar{b}\bar{R} + \bar{c}\bar{R} = \bar{R}$ there exists idempotent $\bar{e} = e + aR$ such that $\bar{b}\bar{e} + \bar{c}$ is an invertible element of $\bar{R}$. By Proposition we have that $aR + (be+c)R = R$.

Let $aR \neq eR = dR$, then $a = da_0$, $e = de_0$ and $a_0R + e_0R = R$ i.e. $a_0u + e_0v = 1$ for some elements $d, a_0, e_0, u, v \in R$. Since $\bar{e}^2 = \bar{e}$ we have $e - e^2 = at$ for some element $t \in R$. Then $e(1-e) = de_0(1-e) = da_0t$ and since

$d \neq 0$ we have $e_0(1-e) = a_0 t$. Since $a_0 u + e_0 v = 1$ we have

$$1-e = e_0(1-e)u + a_0(1-e)v = a_0 tu + a_0(1-e)v = a_0(tu + (1-e)v),$$

i.e. we have $1-e = a_0 k$, where $k = tu + (1-e)v$. So we proved that $a = a_0 d$ where $a_0 R + eR = R$ and $eR \subset dR$ i.e. we have aAe where $r = a_0$, $s = d$ according to the definition of the condition of the adequate of element a to the element e. Theorem is proved.

Corollary. Commutative elementary divisor domain R which is not a ring of stable range 1 exist nonzero and nonunit elements with almost stable range 1.

P r o o f. By Theorem 5 there exists nonunit and nonzero element $a \in R$ and for any $b, c \in R$ such that $aR + bR + cR = R$ there exists element $t \in R$ such that $aR + (bt + c)R = R$. By Proposition a is element with almost stable range 1.

We will notice in the ring of stable range 1 any nonzero and nonunit element is an element with almost stable range 1[2].◊

Recall, that a commutative Bezout rings stable range 1 is an elementary divisor rings [3]. Note, that a commutative J-Noetherian Bezout domain which is not a ring of stable range 1 always contain nonzero and nonunit element with almost stable range 1 which is adequate element of this ring.

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АДЕКВАТНЫЕ СВОЙСТВА ЭЛЕМЕНТОВ ПОЧТИ СТАБИЛЬНОГО РАНГА 1 КОММУТАТИВНОЙ ОБЛАСТИ ЭЛЕМЕНТАРНЫХ ДЕЛИТЕЛЕЙ

Показано, что в коммутативной области элементарных делителей, которая не является кольцом стабильного ранга 1, существуют ненулевые и необратимые элементы почти стабильного ранга 1.

АДЕКВАТНІ ВЛАСТИВОСТІ ЕЛЕМЕНТІВ МАЙЖЕ СТАБІЛЬНОГО РАНГУ 1 КОМУТАТИВНОЇ ОБЛАСТІ ЕЛЕМЕНТАРНИХ ДІЛЬНИКІВ.

Показано, що в комутативній області елементарних дільників, яка не є кільцем стабільного рангу 1, існують ненульові та необоротні елементи майже стабільного рангу 1.

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