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LOCALLY NILPOTENT LIE ALGEBRAS OF DERIVATIONS OF INTEGRAL DOMAINS

Let K be a field of characteristic zero and A an integral domain over K. The Lie algebra $\operatorname{Der}_K A$ of all K-derivations of A carries very important information about the algebra A. This Lie algebra is embedded into the Lie algebra $R\operatorname{Der}_K A \subseteq \operatorname{Der}_K R$, where $R = \operatorname{Frac}(A)$ is the fraction field of A. The rank $\operatorname{rk}_R L$ of a subalgebra L of $R\operatorname{Der}_K A$ is defined as dimension $\dim_R RL$. We prove that every locally nilpotent subalgebra L of $R\operatorname{Der}_K A$ with $\operatorname{rk}_R L = n$ has a series of ideals $0 = L_0 \subset L_1 \subset L_2 \ldots \subset L_n = L$ such that $\operatorname{rk}_R L_i = i$ and all the quotient Lie algebras L_{i+1} / L_i, $i = 0, \mathbf{K}, n-1$, are abelian. We also describe all maximal (with respect to inclusion) locally nilpotent subalgebras L of the Lie algebra $R\operatorname{Der}_K A$ with $\operatorname{rk}_R L = 3$.

Introduction. Let K be a field of characteristic zero and A an associative-commutative algebra over K that is an integral domain. The set of all K-derivations of A forms a Lie algebra $\operatorname{Der}_K A$, which carries important (and often exhaustive) information about the algebra A (see, for example, [8]). In the case of the formal power series ring $A = R[[x_1, x_2, \ldots, x_n]]$, the structure of subalgebras of the Lie algebra $\operatorname{Der}_K A$ is closely connected with the structure of the symmetry groups of differential equations. Finite-dimensional subalgebras of the Lie algebra $\operatorname{Der}_K A$, where $A = K[[x]]$, $A = K[[x, y]]$ and K is the field of real or complex numbers, are described in [2–4].

Each derivation $D \in \operatorname{Der}_K A$ can be uniquely extended to a derivation of the fraction field $R = \operatorname{Frac}(A)$ of A, and if $r \in R$ then one can define a derivation $rD : R \to R$ by setting $rD(x) = r \cdot D(x)$ for all $x \in R$. For the study of the Lie algebra $\operatorname{Der}_K A$, it is convenient to consider a larger Lie algebra $R\operatorname{Der}_K A$. It is an R-linear hull of the set $\{rD \mid r \in R, D \in \operatorname{Der}_K A\}$ and simultaneously a subalgebra (over K) of the Lie algebra $\operatorname{Der}_K R$ of all derivations of R. We will denote the Lie algebra $R\operatorname{Der}_K A$ by $W(A)$. For a subalgebra L of $W(A)$ we define the rank $\operatorname{rk}_R L$ of L over R as $\operatorname{rk}_R L = \dim_R RL$. In [5], nilpotent and solvable subalgebras of finite rank of the Lie algebra $W(A)$ were studied. The structure of nilpotent Lie algebras of derivations with rank 3 was described in [6]. Nilpotent subalgebras of $W(A)$ with the center of large rank were characterized in [9].

In this paper, we study locally nilpotent subalgebras L of the Lie algebra $W(A)$ with $\operatorname{rk}_R L = n$ over the fraction field R. In particular, we prove in Theorem 1 that L contains a series of ideals

$$0 = L_0 \subset L_1 \subset L_2 \ldots \subset L_n = L$$

such that $\operatorname{rk}_R L_i = i$ and all the quotient Lie algebras L_{i+1} / L_i are abelian for $i = 0, 1, \ldots, n-1$. Theorem 2 describes maximal (with respect to inclusion) locally nilpotent subalgebras of rank 3 of the Lie algebra $W(A)$. Note that subalgebras of rank 1 in $W(A)$ are one-dimensional over their field of constants. Lemma 10 describes the structure of subalgebras of rank 2 from $W(A)$ obtained in [7]. The Lie algebras $u_n(K)$ of triangular derivations of polynomial rings, which were investigated in [1], may be the reference point for the study of locally nilpotent Lie algebras of derivations.

We use standard notation. The ground field K is arbitrary of characteristic zero. By R we denote the fraction field of the integral domain A. The Lie algebra

$$R\,\mathrm{Der}_K A = R\langle rD \mid r \in R, D \in \mathrm{Der}_K A\rangle$$

is denoted by $W(A)$. A K-linear hull of elements $x_1, x_2, \dots, x_n$ we write by $K\langle x_1, x_2, \dots, x_n\rangle$. Let L be a subalgebra of $W(A)$. Then the subfield $F = F(L)$ of the field R that consists of all $r \in R$ with $D(r) = 0$ for all $D \in L$ is called the field of constants for L. The rank $\mathrm{rk}_R L$ of L over R is defined by $\mathrm{rk}_R L := \dim_R RL$, where $RL = R\langle rD \mid r \in R, D \in L\rangle$. If I is an ideal of L such that $I = RI \cap L$, then one can define the rank (over R) of the quotient Lie algebra L/I as $\mathrm{rk}_R L/I := \dim_R RL/RI$. By $u_n(K)$ we denote the Lie algebra of all triangular derivations of the polynomial ring $K[x_1, x_2, \dots, x_n]$. This algebra consists of all derivations of the form

$$D = f_1(x_2, x_3, \dots, x_n)\frac{\partial}{\partial x_1} + f_2(x_3, x_4, \dots, x_n)\frac{\partial}{\partial x_2} + \dots + f_n\frac{\partial}{\partial x_n},$$

where $f_i \in K[x_{i+1}, \dots, x_n]$, $i = 1, 2, \dots, n-1$, and $f_n \in K$. A Lie algebra is called locally nilpotent if every its finitely generated subalgebra is nilpotent. The Lie algebra $u_n(K)$ is locally nilpotent but not nilpotent. It contains a series of ideals $0 = I_0 \subset I_1 \subset \dots \subset I_n = u_n(K)$ with abelian factors and $\mathrm{rk}_R I_s = s$ for all $s = 0, 1, \dots, n$ (see [1]). Let V be a vector space over K (not necessary finite dimensional) and T a linear operator on V. The operator T is called locally nilpotent if for each $v \in V$ there exists a number $n = n(v) \geq 1$ such that $T^n(v) = 0$.

On series of ideals in locally nilpotent Lie algebras of derivations. Some auxiliary results are presented in the following lemmas.

Lemma 1. [5, Lemma 1] Let $D_1, D_2 \in W(A)$ and $a, b \in R$. Then

$$[aD_1, bD_2] = ab[D_1, D_2] + aD_1(b)D_2 - bD_2(a)D_1.$$

As we mentioned above, the set RL is an R-linear hull of elements rD for all $r \in R$ and $D \in L$. Analogously we define the set FL for the field of constants $F = F(L)$.

Lemma 2. Let L be a subalgebra of the Lie algebra $W(A)$ and F the field of constants for L. Then:

1. [5, Lemma 2] FL and RL are K-subalgebras of the Lie algebra $W(A)$. Moreover, FL is a Lie algebra over F, and if L is abelian, nilpotent or solvable, then FL has the same property respectively.
2. [5, Lemma 4] If I is an ideal of the Lie algebra L, then the vector space $RI \cap L$ over K is also an ideal of L.
3. [5, Theorem 1] If L is a nilpotent subalgebra of $W(A)$ of finite rank over R, then the Lie algebra FL is finite-dimensional over the field of constants F.
4. [5, Proposition 1] Let L be a nilpotent subalgebra of $W(A)$. If $\mathrm{rk}_R L = 1$, then L is abelian and $\dim_F FL = 1$. If $\mathrm{rk}_R L = 2$ and $\dim_K L \geq 3$,

then there exist $D_1, D_2 \in FL$ and $a \in R$ such that $FL = F\left\langle D_1, aD_1, \ldots, a^k / k! D_1, D_2 \right\rangle$, where $[D_1, D_2] = 0$ and $D_1(a) = 0$, $D_2(a) = 1$.

Lemma 3. [7, Lemmas 5, 8] Let L be a locally nilpotent subalgebra of rank n over R of the Lie algebra $W(A)$ and F the field of constants for L. Then

1. The Lie algebra FL over F is locally nilpotent and $\mathrm{rk}_R FL = n$.
2. If the derived Lie algebra $L' = [L, L]$ is of rank k over R, then $M = RL' \cap L$ is an ideal of L such that $\mathrm{rk}_R M = \mathrm{rk}_R L'$ and FL / FM is an abelian Lie algebra with $\dim_F (FL / FM) = n - k$.

Lemma 4. [5, Lemma 5] Let L be a nilpotent subalgebra of rank $n > 0$ over R of the Lie algebra $W(A)$ and F the field of constants for L. Then L contains a series of ideals $0 = I_0 \subset I_1 \subset \ldots \subset I_{n-1} \subset I_n = L$ such that $\mathrm{rk}_R I_s = s$ and $[I_s, I_s] \subseteq I_{s-1}$ for all $s = 1, \ldots, n$. Moreover, $\dim_F (FL / FI_{n-1}) = 1$.

Lemma 5. Let L be a locally nilpotent subalgebra of the Lie algebra $W(A)$. Let $L_1 \subseteq L_2$ be subalgebras of L such that $L_1 = RL_1 \cap L_2$ is an ideal of L_2. If $\mathrm{rk}_R (L_2 / L_1) = 1$, then L_2 / L_1 is an abelian quotient Lie algebra.

Proof. Let $D + L_1$ be a nonzero element of L_2 / L_1. Then each element of L_2 / L_1 is of the form $rD + L_1$ for some $r \in R$. The elements D and rD generate a nilpotent subalgebra $L_3 = K\langle D, rD \rangle$ of the Lie algebra L since L is locally nilpotent. Every nilpotent subalgebra of rank 1 over R from $W(A)$ is abelian (Lemma 2 (4)). Thus L_3 is an abelian Lie algebra. Then

$$[D + L_1, rD + L_1] = [D, rD] + L_1 \subseteq L_1.$$

Since D, rD are arbitrarily chosen, $[L_2 / L_1, L_2 / L_1] \subseteq L_1$ and the quotient Lie algebra L_2 / L_1 is abelian.¤

Remark 1. Let L be a subalgebra of finite rank over R of the Lie algebra $W(A)$ and I a proper ideal of L such that $I = RI \cap L$. Then $\mathrm{rk}_R L > \mathrm{rk}_R I$.

Lemma 6. Let L be a locally nilpotent subalgebra of finite rank over R of the Lie algebra $W(A)$. Let I be an ideal of L such that $I = RI \cap L$. If the quotient Lie algebra L / I is nonzero, then $\mathrm{rk}_R (L / I)' < \mathrm{rk}_R (L / I)$.

Proof. Suppose to the contrary that there exist a subalgebra L of $W(A)$ and an ideal I of L that satisfy the conditions of the lemma, and $\mathrm{rk}_R (L / I)' = \mathrm{rk}_R (L / I)$. Then this rank equals $n - k$, where $\mathrm{rk}_R L = n$, $\mathrm{rk}_R I = k$. Choose a basis $\left\{ \overline{D}_1, \overline{D}_2, \ldots, \overline{D}_{n-k} \right\}$ of $(L / I)'$ as the set of vectors over R. Under our assumptions this basis is also a basis of L / I over R. Each $\overline{D}_i \in (L / I)'$ is a sum of some commutators from L / I, that is

$$\overline{D}_i = \sum_{j=1}^{k_i} [\overline{S}_j^{(i)}, \overline{T}_j^{(i)}] = \sum_{j=1}^{k_i} [S_j^{(i)}, T_j^{(i)}] + I, \quad k_i \geq 1, \quad i = 1, 2, \ldots, n - k,$$

for some $\overline{S}_j^{(i)}, \overline{T}_j^{(i)} \in L / I$. Let us denote by N the subalgebra of L generated by representatives $S_j^{(i)}, T_j^{(i)}$ of cosets $\overline{S}_j^{(i)}, \overline{T}_j^{(i)}$, $j = 1, \ldots, k_i$, $i = 1, \ldots, n - k$. Since the Lie algebra L is locally nilpotent, the subalgebra N is nilpotent. Denote $L_1 = N + I$. It is easy to see that

$$\mathrm{rk}_R(L_1/I) = \mathrm{rk}_R(L_1/I)' = n-k = \mathrm{rk}_R(L/I).$$

This implies the equalities $\mathrm{rk}_R\, L_1 = \mathrm{rk}_R\, L_1' = n$. Since $L_1/I = N + I/I$; $N/(N \cap I)$ is a nilpotent Lie algebra, the center $Z(L_1/I)$ is nonzero. Let us choose a nonzero $D_1 + I \in Z(L_1/I)$ and denote $J_1 = R(D_1+I) \cap L_1$. Then J_1 is an ideal of L_1 by Lemma 2 (2). Since $RI \cap L = I$ and $D_1 \notin I$, we get $\mathrm{rk}_R\, J_1 = k+1$. If $k+1 < n$, then we take nonzero $D_2 + J_1 \in Z(L_1/J_1)$ and consider $J_2 = R(D_2+J_1) \cap L_1$. The ideal J_2 of L_1 is of rank $k+2$ over R and $RJ_2 \cap L_1 = J_2$. By a continuation of these arguments, we construct a series of ideals

$$I \subset J_1 \subset \ldots \subset J_{n-k-1} \subset J_{n-k} = L_1$$

of the Lie algebra $L_1 = N + I$. Since the quotient algebra L_1/J_{n-k-1} is of rank 1 over R and nilpotent, it is easy to see that L_1/J_{n-k-1} is abelian (Lemma 5). Then $L_1' \subseteq J_{n-k-1}$, and thus $\mathrm{rk}_R\, L_1' \le n-1$. The latter contradicts the equation $\mathrm{rk}_R\, L_1' = n$. The obtained contradiction shows that $\mathrm{rk}_R(L/I)' < \mathrm{rk}_R(L/I)$. ◻

Lemma 7. [7, Lemma 7] Let V be a nonzero vector space over K (not necessary finite dimensional). Let $T_1, T_2, \ldots, T_k$ be pairwise commuting locally nilpotent operators on V. Then there exists a nonzero $v_0 \in V$ such that $T_1(v_0) = T_2(v_0) = \ldots = T_k(v_0) = 0$.

Lemma 8. Let L be a nonzero locally nilpotent subalgebra of finite rank over R of the Lie algebra $W(A)$. Let I be a proper ideal of L such that $I = RI \cap L$. Then the center of the quotient Lie algebra L/I is nonzero.

Proof. Toward the contradiction, suppose the existence of nonzero subalgebras $L \subseteq W(A)$ with a proper ideal I of L that satisfy the conditions of the lemma and $Z(L/I) = 0$. It is observed in Remark 1 that $\mathrm{rk}_R\, I < \mathrm{rk}_R\, L$. Let us choose among these Lie algebras a Lie algebra L with the least rank $\mathrm{rk}_R\, L/I$. Then $\mathrm{rk}_R\, L/I > 1$ (otherwise, in view of Lemma 5, the Lie algebra L/I is abelian and has the nontrivial center).

Let $\mathrm{rk}_R\, L = n$, $\mathrm{rk}_R\, I = k$. Then $\mathrm{rk}_R(L/I) = n-k$. The derived subalgebra $(L/I)' = L' + I$ of the Lie algebra L/I is of rank less then $\mathrm{rk}_R\, L/I$ by Lemma 6. Set $M = R(L'+I) \cap L$. By Lemma 2 (2) M is an ideal of L, and $\mathrm{rk}_R\, M = \mathrm{rk}_R(L'+I) < n$. It is easy to verify that $\mathrm{rk}_R\, M/I \le \mathrm{rk}_R\, M < \mathrm{rk}_R\, L/I$. The subalgebra M of L is locally nilpotent and thus $Z(M/I) \neq 0$ by our choice of the Lie algebra L. Obviously, $Z(M/I)$ is a (possibly infinite-dimensional) vector space over K. Note that the quotient Lie algebra L/M is abelian and $\dim_F(FL/FM) = n-k$ by Lemma 3 (2), where $F = F(L)$ is the field of constants for L.

Choose $D_1, D_2, \ldots, D_{n-k} \in L$ such that the cosets $D_1 + FM$, $D_2 + FM$, ... , $D_{n-k} + FM$ form a basis of the vector space FL/FM over F. Then linear operators $\mathrm{ad}\, D_1, \mathrm{ad}\, D_2, \ldots, \mathrm{ad}\, D_{n-k}$ are locally nilpotent on the vector space $Z(M/I)$ over K. Observe that $Z(M/I)$ is invariant of these linear operators as a characteristics ideal of the Lie algebra M/I. Since $[\mathrm{ad}\, D_i, \mathrm{ad}\, D_j] = \mathrm{ad}[D_i, D_j]$ and $[D_i, D_j] \in M$, linear operators $\mathrm{ad}\, D_i$, $\mathrm{ad}\, D_j$ pairwise commute on $Z(M/I)$ for $i, j = 1, 2, \ldots, n-k$. By Lemma 7, there exists a nonzero element $D_0 + I \in M/I$ such that

$$\mathrm{ad}\, D_i(D_0 + I) = \mathrm{ad}\, D_i(D_0) + I = 0 + I \quad \text{for all } i = 1, 2, \ldots, n-k.$$

Hence $[FD_i, D_0 + I] \subseteq FI$ for all $i = 1, 2, \dots, n-k$. Moreover, $[M, D_0 + I] \subseteq I$ and $[FM, D_0 + I] \subseteq FI$. Since $FL = FD_1 + FD_2 + \dots + FD_{n-k} + FM$, we obtain $[FL, D_0 + I] \subseteq FI$. The latter states that $D_0 + I \in Z(FL / FI)$. Therefore, in view of the condition $I = RI \cap L$, we get $D_0 \in Z(L / I)$.◻

Corollary 1. (see [7, Theorem 1]). Let L be a nonzero locally nilpotent subalgebra of finite rank over R of the Lie algebra $W(A)$. Then the center of the Lie algebra L is nonzero.

Theorem 1. Let L be a locally nilpotent subalgebra of rank n over R of the Lie algebra $W(A)$. Let F be the field of constants for L. Then

1. L contains a series of ideals

$$0 = L_0 \subset L_1 \subset \dots \subset L_n = L \tag{1}$$

such that $\mathrm{rk}_R L_s = s$ and the quotient Lie algebra L_s / L_{s-1} is abelian for all $s = 1, 2 \dots, n$;

2. There exists a basis $\{D_1, \dots, D_n\}$ of L over R such that

$$L_s = (RD_1 + RD_2 + \dots + RD_s) \cap L, \quad [L, D_s] \subseteq L_{s-1}, \quad s = 1, 2 \dots, n;$$

3. $\dim_F FL / FL_{n-1} = 1$.

Proof. (1)-(2) By Corollary 1, there exists a nonzero $D_1 \in Z(L)$. Set $L_1 = RD_1 \cap L$. By Lemma 2 (2), L_1 is an ideal of L of rank 1 over R. Assume that we have constructed basic elements $D_1, D_2, \dots, D_k$ of L over R such that $L_s = (RD_1 + RD_2 + \dots + RD_s) \cap L$ and $[L, D_s] \subseteq L_{s-1}$ for all $s = 1, 2, \dots, k$. Let us construct D_{k+1}. By Lemma 8 the center $Z(L / L_k)$ is nontrivial, so there exists $D_{k+1} \notin L_k$ such that $D_{k+1} + L_k \in Z(L / L_k)$. Then $[L, D_{k+1}] \subseteq L_k$, and one can easily see that $D_1, \dots, D_k, D_{k+1}$ are linearly independent over R. By Lemma 2 (2), $L_{k+1} = RD_{k+1} \cap L + L_k$ is an ideal of L / L_k. In view of the form of the ideal L_k, we get that $L_{k+1} = (RD_1 + \dots + RD_k + RD_{k+1}) \cap L$ is an ideal of L of rank $k+1$ over R. We construct a series of ideals (1) and a basis from the conditions of the theorem. Moreover, since $\mathrm{rk}_R (L_{s+1} / L_s) = 1$ Lemma 5 implies that the quotient Lie algebras L_{s+1} / L_s are abelian for all $s = 0, 1, \dots, n-1$.

(3) The proof is analogous to the proof of Lemma 4.◻

Locally nilpotent subalgebras of rank 3 of the Lie algebra $W(A)$. In the following lemma the main results of [6] are collected.

Lemma 9. [6, Lemmas 8, 9] Let L be a nilpotent subalgebra of rank 3 over R of the Lie algebra $W(A)$. Let $Z(L)$ be the center of L and F the field of constants for L. If $\dim_F FL \geq 4$, then there exist $a, b \in R$, integers $k \geq 1, n \geq 0, m \geq 1$, and pairwise commuting elements $D_1, D_2, D_3 \in L$ such that the Lie algebra FL is contained in the nilpotent subalgebra $\hat{L} \subseteq W(A)$ of one of the following types:

1. If $\mathrm{rk}_R Z(L) = 2$, then

$$\hat{L} = F\langle D_3, D_1, aD_1, \dots, \frac{a^k}{k!} D_1, D_2, aD_2, \dots, \frac{a^k}{k!} D_2 \rangle,$$

where $D_1(a) = D_2(a) = 0$ and $D_3(a) = 1$.

2. If $\mathrm{rk}_R Z(L) = 1$, then $\hat{L}$ is either the same as in (1), or

$$\tilde{L} = F\langle D_3, D_2, aD_2, \mathbf{K}, \tfrac{a^n}{n!} D_2, \left\{\tfrac{a^i b^j}{i!j!} D_1\right\}_{i,j=0}^{n,m}\rangle,$$

where $D_1(a) = D_2(a) = 0$ and $D_3(a) = 1,$ $D_1(b) = D_3(b) = 0$ and $D_2(b) = 1.$

In [7], the description of locally nilpotent subalgebras of $W(A)$ of ranks 1 and 2 was given.

Lemma 10. [7, Theorem 2] Let L be a locally nilpotent subalgebra of the Lie algebra $W(A)$ and F the field of constants for L.

1. If $\mathrm{rk}_R L = 1$, then L is abelian and $\dim_F FL = 1$.
2. If $\mathrm{rk}_R L = 2$, then FL is either nilpotent finite dimensional over F, or infinite dimensional over F and there exist $D_1, D_2 \in L,$ $a \in R$ such that

$$FL = \langle D_2, D_1, aD_1, \mathbf{K}, \frac{a^k}{k!} D_1, \mathbf{K}\rangle,$$

where $[D_1, D_2] = 0$, $D_1(a) = 0$, and $D_2(a) = 1$.

Theorem 2. Let L be a maximal (with respect to inclusion) locally nilpotent subalgebra of the Lie algebra $W(A)$ such that $\mathrm{rk}_R L = 3$. Let F be the field of constants for L. Then $FL = L$ and L is a Lie algebra over F of one of the following types:

1. L is a nilpotent Lie algebra of dimension 3 over F;
2. $L = F\langle D_3, \left\{\tfrac{a^i}{i!} D_1\right\}_{i=0}^{\infty}, \left\{\tfrac{a^i}{i!} D_2\right\}_{i=0}^{\infty}\rangle,$ where $D_1, D_2, D_3 \in L$ and $a \in R$ such that $D_1(a) = D_2(a) = 0,$ $D_3(a) = 1,$ and $[D_i, D_j] = 0$ for all $i, j = 1, 2, 3;$
3. $L = F\langle D_3, \left\{\tfrac{a^i}{i!} D_2\right\}_{i=0}^{\infty}, \left\{\tfrac{a^i b^j}{i!j!} D_1\right\}_{i,j=0}^{\infty}\rangle,$ where $D_1, D_2, D_3 \in L$ and $a, b \in R$ such that $D_1(a) = D_2(a) = 0,$ $D_3(a) = 1,$ and $D_1(b) = D_3(b) = 0,$ $D_2(b) = 1,$ and $[D_i, D_j] = 0$ for all $i, j = 1, 2, 3.$

Proof. The Lie algebra FL is locally nilpotent by Lemma 3. Therefore, the maximality of the subalgebra $L \subseteq W(A)$ implies $FL = L$. By Corollary 1 $Z(L) \neq 0$. If $\mathrm{rk}_R Z(L) = 3$, then one can easily see that L is abelian. Thus, it follows from Lemma 2 that L is the abelian Lie algebra of dimension 3 over F and L is of type (1) from the conditions of the theorem.

Case 1. Let $\mathrm{rk}_R Z(L) = 2$. Let us choose arbitrary elements $D_1, D_2 \in Z(L)$ linearly independent over R and set $I = (RD_1 + RD_2) \cap L$. Then in view of Theorem 1, I is an ideal of the Lie algebra L of rank 2 over R and $\dim_F(FL / FI) = 1.$ It is easy to verify that I is abelian. Indeed, take an arbitrary $D = r_1 D_1 + r_2 D_2 \in I.$ Then

$$[D_1, D] = D_1(r_1)D_1 + D_1(r_2)D_2 = 0, \quad [D_2, D] = D_2(r_1)D_1 + D_2(r_2)D_2 = 0,$$

whence it follows $r_1, r_2 \in \mathrm{Ker}\, D_1 \cap \mathrm{Ker}\, D_2$. Therefore, for all $D, D' \in I$ we get $[D, D'] = 0.$

Let us take an element $D_3 \in L / I.$ It was proved above that $FL = FI + FD_3.$ Consider a nonabelian finitely generated (over K) subalgebra M of the Lie algebra L such that $D_1, D_2, D_3 \in M.$ Since $\mathrm{rk}_R M = 3$ and $\mathrm{rk}_R Z(M) = 2,$ Lemma 9 implies that FM is contained in some subalgebra L_M of $W(A)$ of the form

$$L_M = F\left\langle D_3, D_1, aD_1, \dots, a^n / n! D_1, D_2, aD_2, \dots, a^n / n! D_2 \right\rangle,$$

where $a \in R$ such that $D_3(a) = 1$ and $D_1(a) = D_2(a) = 0$. Then M is contained in the locally nilpotent Lie algebra of the form

$$L_1 = F\langle D_3, D_1, aD_1, \dots, a^n / n!D_1, \dots, D_2, aD_2, \dots, a^n / n!D_2, \dots\rangle,$$

where $a \in R$ is defined by derivations D_1, D_2, D_3 up to a summand in F. Since the subalgebra M is an arbitrarily chosen in L, we have $L \subseteq L_1$. In view of the maximality of the Lie algebra L, we get that $L = L_1$ and L is of type (2) from the conditions of the theorem.

Case 2. Let $\mathrm{rk}_R Z(L) = 1$. If $\dim_F FL = 3$, then the Lie algebra L is nilpotent of dimension 3 over F and L is of type (1) from the conditions of the theorem. Therefore, further we assume that $\dim_F FL \geq 4$. Take a nonzero $D_1 \in Z(L)$. Then $I_1 = RD_1 \cap L$ is an ideal of the Lie algebra L of rank 1 over R (by Lemma 2). The center of the quotient Lie algebra L / I_1 is nontrivial by Lemma 8 and thus, we may choose a nonzero $D_2 + I_1 \in Z(L / I_1)$. By Theorem 1, $I_2 = (RD_1 + RD_2) \cap L$ is an ideal of the Lie algebra L, $\mathrm{rk}_R I_2 = 2$, and $\dim_F FL / FI_2 = 1$. Then for some $D_3 \in FL \setminus FI_2$ we get $FL = FI_2 + FD_3$. Moreover, from the choice of D_2 it is easy to see that $[D_3, D_2] \in I_1$, so $[D_3, D_2] = r_3 D_1$ for some $r_3 \in R$. In particular, this implies that derivations D_2 and D_3 are commuting on $\mathrm{Ker}\, D_1$, i. e.

$$D_3(D_2(x)) = D_2(D_3(x)) \text{ for all } x \in \mathrm{Ker}\, D_1.$$

Let us show that the ideal FI_2 is nonabelian. Suppose this is not true and FI_2 is abelian. Since $\dim_F FL \geq 4$, $\dim_F FI_2 \geq 3$. Then there exists $D = r_1 D_1 + r_2 D_2 \in FI_2$ such that at least one of the coefficients r_1, r_2 is not in F. From the obvious equalities

$$[D_1, D] = D_1(r_1)D_1 + D_1(r_2)D_2 = 0, \; [D_2, D] = D_2(r_1)D_1 + D_2(r_2)D_2 = 0,$$

it follows $r_1, r_2 \in \mathrm{Ker}\, D_1 \cap \mathrm{Ker}\, D_2$.

Since at least one of r_1, r_2 not in F, either $D_3(r_1) \neq 0$ or $D_3(r_2) \neq 0$. Firstly, let $D_3(r_2) \neq 0$. The relation $[D_3, D_2] = r_3 D_1$ implies that for any integer $m \geq 1$ it holds

$$(\mathrm{ad}\, D_3)^m (D) = R_m D_1 + D_3^m (r_2) D_2 \text{ for some } R_m \in R.$$

Since the linear operator $\mathrm{ad}\, D_3$ is locally nilpotent on L, there exists an integer $k > 1$ such that

$$D_3^{k-1}(r_2) \neq 0, \; D_3^k (r_2) = 0.$$

Let us denote $r_0 = D_3^{k-2}(r_2)$. Then $D_3(r_0) \neq 0$ and $D_3^2(r_0) = 0$. Furthermore, it is easy to verify that $D_1(r_0) = D_2(r_0) = 0$. Set $a = \frac{r_0}{D_3(r_0)} \in R \setminus F$. One can easily checked that $D_1(a) = D_2(a) = 0$ and $D_3(a) = 1$.

Now let $D_3(r_2) = 0$. Then $r_2 \in F$, so $r_1 \notin F$ and $r_1 D_1 \in FI_2$ (because $r_2 D_2 \in FI_2$ and $D \in FI_2$). Note that $D_3(r_1) \neq 0$. Using the relation

$$(\mathrm{ad}\, D_3)^m (r_1 D_1) = D_3^m (r_1) D_1, \; m \geq 1,$$

one can show (as in the case $r_2 \notin F$) that there exists $a \in R$ such that $D_1(a) = D_2(a) = 0$ and $D_3(a) = 1$.

Let us prove that $FI_1 = F[a]D_1$, where $F[a] = \{ f(a) \mid f(t) \in F[t] \}$. Consider the sum $L_1 = L + F[a]D_1$. Since $[I_2, F[a]D_1] = 0$ and $[D_3, F[a]D_1] \subseteq F[a]D_1$, L_1 is a subalgebra of $W(A)$ and $L \subseteq L_1$. From the maximality of L, we get $L = L_1$.

Hence $F[a]D_1 \subseteq FI_1$. Conversely, take an arbitrary element $rD_1 \in I_1$. Then $D_1(r) = 0$ and $D_2(r) = 0$ since the ideal FI_2 is abelian by our assumption. The operator ad D_3 acts locally nilpotently on rD_1, so $D_3^k(r) = 0$ for some integer $k \geq 1$. One can show (using [7, Lemma 6]) that r is a linear combination over the field F of elements $1, a, \ldots, a^t$ for some positive integer t. Thus $rD_1 \in F[a]D_1$. Therefore, $I_1 \subseteq F[a]D_1$ and $FI_1 = F[a]D_1$.

Since $[D_3, D_2] \in I_1$, we get that $[D_3, D_2] = f(a)D_1$ for some $f(t) \in F[t]$. The field F is of characteristic zero, so there exists a polynomial $g(t) \in F[t]$ such that $g'(t) = f(t)$. Note that $D_3(g(a)) = f(a)$ since $D_3(a) = 1$. Set $\mathring{D}_2 = D_2 - g(a)D_1 \in I_2$. Then $[D_3, \mathring{D}_2] = 0$, and $\mathring{D}_2$ has the same other properties as the derivation D_2. Thus we may assume without loss of generality that $[D_3, D_2] = 0$. Then $D_2 \in Z(L)$ and $\operatorname{rk}_R Z(L) = 2$, which contradicts our assumption. This means that the ideal FI_2 of the Lie algebra FL is nonabelian.

Let us show that L is a Lie algebra of type (3) from the conditions of the theorem. Consider an arbitrary nonabelian finitely generated subalgebra M of FI_2 such that $D_1, D_2 \in M$ (such a subalgebra exists because FI_2 is a nonabelian ideal of L). Denote by N a subalgebra of L generated by the Lie algebra M and D_3. Then $\operatorname{rk}_R N = 3$ and N is a nonabelian Lie algebra that contains a nonabelian ideal N_2 of rank 2 over R such that $\dim_F FN / FN_2 = 1$. By Lemma 9, there exist $a, b \in R$ such that $D_1(a) = D_2(a) = 0$, $D_3(a) = 1$, $D_1(b) = D_3(b) = 0$, and $D_2(b) = 1$ and FN is contained in the Lie algebra L_N of the form

$$L_N = F\langle D_3, D_2, aD_2, \ldots, \frac{a^k}{k!} D_2, \left\{ \frac{a^i b^j}{i! j!} D_1 \right\}_{i,j=0}^{k,m} \rangle.$$

The elements $a, b \in R$ are uniquely determined by the derivations D_1, D_2, D_3 up to a summand in F. Thus N is contained in the Lie algebra

$$L_2 = F\langle D_3, D_2, aD_2, \ldots, \frac{a^k}{k!} D_2, \ldots, \left\{ \frac{a^i b^j}{i! j!} D_1 \right\}_{i,j=0}^{\infty} \rangle.$$

The Lie algebra L_2 is a locally nilpotent subalgebra of $W(A)$ of rank 3 over R. Since N is an arbitrarily chosen subalgebra of L, L is contained in L_2. In view of maximality of the Lie algebra L, we obtain $L = L_2$. The proof is complete.◻

Example 1. Let $A = K[x_1, x_2, x_3]$ and $R = K(x_1, x_2, x_3)$. Then the Lie algebra $L = K\langle x_1 \frac{\partial}{\partial x_1}, x_2 \frac{\partial}{\partial x_2}, x_3 \frac{\partial}{\partial x_3} \rangle$ is abelian, $\operatorname{rk}_R L = 3$, and L is a maximal locally nilpotent subalgebra of the Lie algebra $W_3(K)$ (see [9, Proposition 1]).

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ОБ ЛОКАЛЬНО НИЛЬПОТЕНТНЫХ АЛГЕБРАХ ЛИ ДИФФЕРЕНЦИРОВАНИЙ ОБЛАСТЕЙ ЦЕЛОСТНОСТИ

Пусть K *– поле характеристики ноль и* A *– область целостности над* K. *Алгебра Ли* $\mathrm{Der}_K A$ *всех* K-*дифференцирований* A *несет очень важную информацию об алгебре* A. *Эта алгебра Ли вкладывается в алгебру Ли* $R\,\mathrm{Der}_K A \subseteq \mathrm{Der}_K R$, *где* $R = \mathrm{Frac}(A)$ *– поле частных над* A. *Ранг* $\mathrm{rk}_R L$ *подалгебры* L *из* $R\,\mathrm{Der}_K A$ *определяется как размерность* $\dim_R RL$. *Доказано, что каждая локально нильпотентная подалгебра* L *из* $R\,\mathrm{Der}_K A$ *с рангом* $\mathrm{rk}_R L = n$ *содержит ряд идеалов* $0 = L_0 \subset L_1 \subset L_2 \ldots \subset L_n = L$ *такой, что* $\mathrm{rk}_R L_i = i$ *и все фактор-алгебры Ли* L_{i+1} / L_i, $i = 0, \mathbf{K}, n-1$, *абелевы. Также описаны все максимальные (относительно включения) локально нильпотентные подалгебры* L *из алгебры Ли* $R\,\mathrm{Der}_K A$, *в которых* $\mathrm{rk}_R L = 3$.

ПРО ЛОКАЛЬНО НІЛЬПОТЕНТНІ АЛГЕБРИ ЛІ ДИФЕРЕНЦІЮВАНЬ ОБЛАСТЕЙ ЦІЛІСНОСТІ

Нехай K *– поле характеристики нуль і* A *– область цілісності над* K. *Алгебра Лі* $\mathrm{Der}_K A$ *всіх* K-*диференціювань* A *несе дуже важливу інформацію про алгебру* A. *Ця алгебра Лі вкладається в алгебру Лі* $R\,\mathrm{Der}_K A \subseteq \mathrm{Der}_K R$, *де* $R = \mathrm{Frac}(A)$ *– це поле часток над* A. *Ранг* $\mathrm{rk}_R L$ *підалгебри* L *з* $R\,\mathrm{Der}_K A$ *визначається як розмірність* $\dim_R RL$. *Доведено, що кожна локально нільпотентна підалгебра* L *з* $R\,\mathrm{Der}_K A$ *з рангом* $\mathrm{rk}_R L = n$ *містить ряд ідеалів* $0 = L_0 \subset L_1 \subset L_2 \ldots \subset L_n = L$ *такий, що* $\mathrm{rk}_R L_i = i$ *і всі фактор-алгебри Лі* L_{i+1} / L_i, $i = 0, \mathbf{K}, n-1$, *абелеві. Також описані всі максимальні (за включенням) локально нільпотентні підалгебри* L *з алгебри Лі* $R\,\mathrm{Der}_K A$, *в яких* $\mathrm{rk}_R L = 3$.

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